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Geometrical Progression

GEOMETRICAL PROGRESSION (GP)

A sequence $a_1, a_2, a_3, \ldots, a_n$ *is called a geometrical progression, if each term is nonzero and* $\frac{a_{k+1}}{a_k} = r$ *(constant) for all* $k \geq 1$.

The constant ratio is called its *common ratio*.

A geometrical progression is abbreviated as GP.

In a GP we usually denote the *first term* by *a*, the *common ratio* by *r* and the *nth term* by T_n.

The *n*th term of a GP is called its general term.

SOME EXAMPLES OF GP

EXAMPLE 1 *Show that the progression* $6, 18, 54, 162, \ldots$ *is a GP. Write down its first term and the common ratio.*

SOLUTION We have $\frac{18}{6} = \frac{54}{18} = \frac{162}{54} = 3$ (constant).

So, the given progression is a GP in which the first term = 6 and the common ratio = 3.

EXAMPLE 2 *Show that the progression* $-16, 4, -1, \frac{1}{4}, \cdots$ *is a GP. Write down its first term and the common ratio.*

SOLUTION We have $\frac{4}{-16} = \frac{-1}{4} = \frac{(1/4)}{-1} = \frac{-1}{4}$ (constant).

So, the given progression is a GP in which $a = -16$ and $r = \frac{-1}{4}$.

EXAMPLE 3 *Show that the progression* $\frac{1}{2}, \frac{-1}{3}, \frac{2}{9}, \frac{-4}{27}, \cdots$ *is a GP. Write down its first term and find the common ratio.*

SOLUTION We have

$$\left(\frac{-1}{3}\right) \div \frac{1}{2} = \left(\frac{-1}{3} \times 2\right) = \frac{-2}{3}, \quad \frac{2}{9} \div \left(\frac{-1}{3}\right) = \left\{\frac{2}{9} \times \frac{3}{(-1)}\right\} = \frac{-2}{3},$$

$$\left(\frac{-4}{27}\right) \div \frac{2}{9} = \left(\frac{-4}{27} \times \frac{9}{2}\right) = \frac{-2}{3}.$$

So, the given progression is a GP in which $a = \frac{1}{2}$ and $r = \frac{-2}{3}$.

EXAMPLE 4 *Show that the sequence given by* $T_n = (2 \times 3^n)$ *for all* $n \in N$ *is a GP. Find its first term and the common ratio.*

SOLUTION We have

$$T_n = (2 \times 3^n) \text{ and } T_{n+1} = (2 \times 3^{n+1}).$$

$$\therefore \quad \frac{T_{n+1}}{T_n} = \frac{2 \times 3^{n+1}}{2 \times 3^n} = 3 \text{ (constant) for all } n \in N.$$

Also, $T_1 = (2 \times 3^1) = (2 \times 3) = 6.$

$\therefore$ the first term = 6 and the common ratio = 3.

GEOMETRIC SERIES *If* $a_1, a_2, a_3, \ldots, a_n, \ldots$ *is a GP then the sum* $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ *is called a geometric series.*

A geometric series is finite or infinite according as the corresponding GP is finite or infinite.

GENERAL TERM OF A GP

THEOREM 1 *Prove that the nth term of a GP with first term a and common ratio r is given by* $T_n = ar^{n-1}$.

PROOF Let us consider a GP in which first term = a and common ratio = r. Then, the GP is $a, ar, ar^2, \ldots$.

Its first term, $T_1 = a = ar^{(1-1)}$;

second term, $T_2 = ar = ar^{(2-1)}$;

third term, $T_3 = ar^2 = ar^{(3-1)}$;

… … … … …

… … … … …

$\therefore$ nth term, $T_n = ar^{(n-1)}$.

Hence, $T_n = ar^{n-1}$ (known as the general term of the GP).

REMEMBER

In a GP with first term = a and common ratio = r, we have

nth term, $T_n = ar^{n-1}$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the* 10*th term and the general term of the progression*

$$\frac{1}{4}, \frac{-1}{2}, 1, -2, 4, \ldots .$$

SOLUTION In the given progression, we have

$$\left(\frac{-1}{2}\right) \div \frac{1}{4} = \left(\frac{-1}{2} \times 4\right) = -2, \quad 1 \div \left(\frac{-1}{2}\right) = 1 \times (-2) = -2,$$

$$(-2) \div 1 = -2 \text{ and } 4 \div (-2) = -2.$$

So, the given progression is a GP in which $a = \frac{1}{4}$ and $r = -2$.

$\therefore$ the 10th term, $T_{10} = ar^{(10-1)} = ar^9 = \frac{1}{4} \times (-2)^9 = \frac{-512}{4} = -128.$

The general term, $T_n = ar^{(n-1)} = \frac{1}{4} \times (-2)^{(n-1)} = (-1)^{(n-1)} \times 2^{(n-3)}.$

EXAMPLE 2 *Show that the progression*

$$1, \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \frac{(3-2\sqrt{2})}{12}, \frac{(5\sqrt{2}-7)}{24\sqrt{3}}, \ldots$$

is a GP. Find its 5th term.

SOLUTION In the given progression, we have

$$\frac{T_2}{T_1} = \left\{\frac{(\sqrt{2}-1)}{2\sqrt{3}} \times \frac{1}{1}\right\} = \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \frac{T_3}{T_2} = \frac{(3-2\sqrt{2})}{12} \times \frac{2\sqrt{3}}{(\sqrt{2}-1)} = \frac{(\sqrt{2}-1)}{2\sqrt{3}},$$

$$\frac{T_4}{T_3} = \frac{(5\sqrt{2}-7)}{24\sqrt{3}} \times \frac{12}{(3-2\sqrt{2})} = \frac{(\sqrt{2}-1)}{2\sqrt{3}}.$$

$\therefore$ $\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \ldots = \frac{(\sqrt{2}-1)}{2\sqrt{3}}$ (constant).

So, the given progression is a GP in which $a = 1$ and $r = \frac{(\sqrt{2}-1)}{2\sqrt{3}}.$

$\therefore$ the 5th term, $T_5 = ar^{(5-1)} = ar^4 = 1 \times \left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)^4$

$$= \frac{(\sqrt{2}-1)^4}{144} = \frac{(3-2\sqrt{2})^2}{144}$$

$$= \frac{(17-12\sqrt{2})}{144}.$$

Hence, $T_5 = \frac{(17-12\sqrt{2})}{144}.$

EXAMPLE 3 *Which term of the GP* 2, 8, 32, 128, … *is* 131072*?*

SOLUTION In the given GP, we have $a = 2$ and $r = \frac{8}{2} = 4.$

Let its *n*th term be 131072. Then,

$$T_n = 131072 \Rightarrow ar^{(n-1)} = 131072$$

$$\Rightarrow 2 \times 4^{(n-1)} = 131072$$

$$\Rightarrow 4^{(n-1)} = 65536 = 4^8$$

$$\Rightarrow n - 1 = 8 \Rightarrow n = 9.$$

Hence, the 9th term of the given GP is 131072.

EXAMPLE 4 *Which term of the GP* $\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{1}{9\sqrt{3}}, \ldots$ *is* $\frac{1}{729\sqrt{3}}$*?*

SOLUTION In the given GP, we have $a = \sqrt{3}$ and $r = \left(\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}}\right) = \frac{1}{3}.$

Let its nth term be $\frac{1}{729\sqrt{3}}$. Then,

$$T_n = \frac{1}{729\sqrt{3}} \Rightarrow ar^{(n-1)} = \frac{1}{729\sqrt{3}}$$

$$\Rightarrow \sqrt{3} \times \left(\frac{1}{3}\right)^{(n-1)} = \frac{1}{729\sqrt{3}}$$

$$\Rightarrow \frac{1}{3^{(n-1)}} = \frac{1}{(729 \times 3)} = \frac{1}{2187} = \frac{1}{3^7}$$

$$\Rightarrow n - 1 = 7 \Rightarrow n = 8.$$

Hence, the 8th term of the given GP is $\frac{1}{729\sqrt{3}}$.

EXAMPLE 5 *If the 4th and 9th terms of a GP are 54 and 13122 respectively, find the GP. Also, find its general term.*

SOLUTION Let a be the first term and r be the common ratio of the given GP.

Then, $T_4 = 54 \Rightarrow ar^{(4-1)} = 54 \Rightarrow ar^3 = 54.$ … (i)

And, $T_9 = 13122 \Rightarrow ar^{(9-1)} = 13122 \Rightarrow ar^8 = 13122.$ … (ii)

On dividing (ii) by (i), we get

$$\frac{ar^8}{ar^3} = \frac{13122}{54} \Rightarrow r^5 = 243 = 3^5 \Rightarrow r = 3.$$

Putting $r = 3$ in (i), we get

$$a \times 3^3 = 54 \Rightarrow 27a = 54 \Rightarrow a = 2.$$

Thus, $a = 2$ and $r = 3$.

So, the required GP is 2, 6, 18, 54, … .

The general term of the GP is given by

$$T_n = ar^{(n-1)} = \{2 \times 3^{(n-1)}\}.$$

EXAMPLE 6 *The first term of a GP is 1 and the sum of its 3rd and 5th terms is 90. Find the common ratio of the GP.*

SOLUTION It is given that the first term of the GP is 1. Let r be the common ratio of this GP. Then,

$$T_3 = 1 \times r^{(3-1)} = r^2 \text{ and } T_5 = 1 \times r^{(5-1)} = r^4.$$

$$\therefore \quad (T_3 + T_5) = 90 \Rightarrow r^2 + r^4 = 90$$

$$\Rightarrow r^4 + r^2 - 90 = 0$$

$$\Rightarrow t^2 + t - 90 = 0, \text{ where } r^2 = t$$

$$\Rightarrow (t + 10)(t - 9) = 0$$

$$\Rightarrow t = -10 \text{ or } t = 9$$

$$\Rightarrow t = 9 \qquad [\because\ t = r^2 \neq -10]$$

$$\Rightarrow r^2 = 9$$

$$\Rightarrow r = 3 \text{ or } r = -3.$$

Hence, the common ratio of the given GP is 3 or –3.

EXAMPLE 7 *The 4th, 7th and 10th terms of a GP are a, b, c respectively. Prove that* $b^2 = ac$.

SOLUTION Let *A* be the first term and *r* be the common ratio of the given GP. Then,

$$a = Ar^{(4-1)} = Ar^3;\ b = Ar^{(7-1)} = Ar^6 \text{ and } c = Ar^{(10-1)} = Ar^9.$$

$$\therefore \quad ac = (Ar^3) \times (Ar^9) = A^2r^{12} = (Ar^6)^2 = b^2.$$

Hence, $b^2 = ac$.

EXAMPLE 8 *If a, b, c are three consecutive terms of an AP and x, y, z are three consecutive terms of a GP, then prove that*

$$x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)} = 1.$$

SOLUTION It is given that *a, b, c* are in AP. Let *d* be the common difference of this AP. Then,

$$(b-c) = -(c-b) = -d,\ (c-a) = \{(c-b) + (b-a)\} = 2d$$

and $(a-b) = -(b-a) = -d$.

Also, *x, y, z* are in GP. So, $y = \sqrt{xz}$.

$$\therefore \quad x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)}$$

$$= x^{-d} \times (\sqrt{xz})^{2d} \times z^{-d}$$

$$[\because\ (b-c) = -d, (c-a) = 2d, (a-b) = -d \text{ and } y = \sqrt{xz}]$$

$$= x^{-d} \times (xz)^d \times z^{-d}$$

$$= x^{-d} \times x^d \times z^d \times z^{-d} = x^{(-d+d)} \times z^{(-d+d)} = (x^0 \times z^0) = 1.$$

Hence, $x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)} = 1$.

EXAMPLE 9 *If a, b, c, d are in GP, prove that*

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2.$$

SOLUTION Let *r* be the common ratio of the GP *a, b, c, d*.

Then, $b = ar, c = ar^2$ and $d = ar^3$.

$$\therefore \quad \text{LHS} = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$$

$$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$$

$$= a^4r^2(1 + r^2 + r^4)^2.$$

$$\text{And, RHS} = (ab + bc + cd)^2 = (a^2r + a^2r^3 + a^2r^5)^2$$

$$= a^4r^2(1 + r^2 + r^4)^2.$$

Hence, $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

EXAMPLE 10 *If a, b, c, d are in GP, prove that*

$$(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2.$$

SOLUTION Let *r* be the common ratio of the GP *a, b, c, d*.

Then, $b = ar, c = ar^2$ and $d = ar^3$.

$$\therefore \quad \text{LHS} = (b-c)^2 + (c-a)^2 + (d-b)^2$$

$$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$$

$$= \{a(r - r^2)\}^2 + \{a(r^2 - 1)\}^2 + \{a(r^3 - r)\}^2$$
$$= a^2(r - r^2)^2 + a^2(r^2 - 1)^2 + a^2(r^3 - r)^2$$
$$= a^2 \cdot \{(r - r^2)^2 + (r^2 - 1)^2 + (r^3 - r)^2\}$$
$$= a^2 \cdot \{(r^2 + r^4 - 2r^3) + (r^4 + 1 - 2r^2) + (r^6 + r^2 - 2r^4)\}$$
$$= a^2 \cdot (r^6 - 2r^3 + 1) = a^2(1 - r^3)^2$$
$$= (a - ar^3)^2 = (a - d)^2 = \text{RHS.}$$

Hence, $(b - c)^2 + (c - a)^2 + (d - b)^2 = (a - d)^2$.

EXAMPLE 11 *If the pth, qth and rth terms of a GP be a, b, c respectively then prove that*

$$a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)} = 1.$$

SOLUTION Let *A* be the first term and *R* be the common ratio of the given GP. Then,

$$T_p = a \Rightarrow a = AR^{(p-1)}$$
$$T_q = b \Rightarrow b = AR^{(q-1)}$$
$$T_r = c \Rightarrow c = AR^{(r-1)}.$$

$\therefore$ $a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)}$

$$= \{AR^{(p-1)}\}^{(q-r)} \cdot \{AR^{(q-1)}\}^{(r-p)} \cdot \{AR^{(r-1)}\}^{(p-q)}$$
$$= A^{(q-r)} \cdot R^{(p-1)(q-r)} \cdot A^{(r-p)} \cdot R^{(q-1)(r-p)} \cdot A^{(p-q)} \cdot R^{(r-1)(p-q)}$$
$$= A^{\{(q-r)+(r-p)+(p-q)\}} \cdot R^{\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\}}$$
$$= A^0 \cdot R^{p(q-r)+q(r-p)+r(p-q)+(r-q)+(p-r)+(q-p)}$$
$$= (1 \times R^0) = (1 \times 1) = 1.$$

Hence, $a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)} = 1$.

EXAMPLE 12 *If the first and the nth terms of a GP are a and b respectively and P is the product of its first n terms then prove that* $P^2 = (ab)^n$.

SOLUTION Let *r* be the comon ratio of the given GP. Then,

$$T_n = b \Rightarrow ar^{n-1} = b \Rightarrow r^{n-1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n-1)}} \cdot \quad \ldots \text{(i)}$$

$\therefore$ P = product of first *n* terms of the GP

$$= a \cdot ar \cdot ar^2 \ldots \cdot ar^{n-1}$$
$$= a^n \cdot r^{\{1+2+3+\ldots+(n-1)\}} = a^n \cdot r^{\frac{1}{2}n(n-1)}$$

[taking the sum of the AP]

$$= a^n \cdot \left\{\left(\frac{b}{a}\right)^{\frac{1}{(n-1)}}\right\}^{\frac{1}{2}n(n-1)}$$

[using (i)]

$$= a^n \cdot \left(\frac{b}{a}\right)^{\left\{\frac{1}{(n-1)} \times \frac{1}{2}n(n-1)\right\}} = a^n \cdot \left(\frac{b}{a}\right)^{\frac{n}{2}}$$

$\Rightarrow$ $P^2 = a^{2n} \cdot \left(\frac{b}{a}\right)^n = \left(a^{2n} \cdot \frac{b^n}{a^n}\right) = a^n b^n = (ab)^n.$

Hence, $P^2 = (ab)^n$.

EXAMPLE 13 *The $(m+n)$th and the $(m-n)$th terms of a GP are p and q respectively. Show that the mth and the nth terms of the GP are $\sqrt{pq}$ and $p\cdot\left(\frac{q}{p}\right)^{(m/2n)}$ respectively.*

SOLUTION Let a be the first term and r be the common ratio of the given GP. Then,

$$T_{m+n} = p \text{ and } T_{m-n} = q$$

$$\Rightarrow \quad ar^{(m+n-1)} = p \quad \ldots \text{(i)} \qquad \text{and} \quad ar^{(m-n-1)} = q \quad \ldots \text{(ii)}$$

$$\Rightarrow \quad \frac{ar^{(m+n-1)}}{ar^{(m-n-1)}} = \frac{p}{q}$$

$$\Rightarrow \quad r^{\{(m+n-1)-(m-n-1)\}} = \frac{p}{q}$$

$$\Rightarrow \quad r^{2n} = \frac{p}{q} \Rightarrow r = \left(\frac{p}{q}\right)^{\frac{1}{2n}}$$

$$\Rightarrow \quad \frac{1}{r} = \left(\frac{q}{p}\right)^{\frac{1}{2n}}. \quad \ldots \text{(iii)}$$

$$\therefore \quad T_m = ar^{(m-1)}$$

$$= ar^{(m+n-1)} \times \left(\frac{1}{r}\right)^n$$

$$= p \times \left\{\left(\frac{q}{p}\right)^{\frac{1}{2n}}\right\}^n \qquad \text{[using (i) and (iii)]}$$

$$= p \times \left(\frac{q}{p}\right)^{\left(\frac{1}{2n}\times n\right)} = p \times \left(\frac{q}{p}\right)^{\frac{1}{2}} = \sqrt{pq}.$$

And, $T_n = ar^{(n-1)}$

$$= ar^{(m+n-1)} \times \left(\frac{1}{r}\right)^m$$

$$= p \times \left\{\left(\frac{q}{p}\right)^{\frac{1}{2n}}\right\}^m = p \times \left(\frac{q}{p}\right)^{\frac{m}{2n}} \qquad \text{[using (i) and (iii)].}$$

Hence, $T_m = \sqrt{pq}$ and $T_n = p \times \left(\frac{q}{p}\right)^{\frac{m}{2n}}$.

EXAMPLE 14 *The pth, qth and rth terms of an AP as well as those of a GP are a, b, c, respectively. Prove that*

$$a^{b-c}\cdot b^{c-a}\cdot c^{a-b} = 1.$$

SOLUTION Let x be the first term and d be the common difference of the AP. Then,

$$a = x+(p-1)d,\ b = x+(q-1)d \text{ and } c = x+(r-1)d.$$

$$\therefore \quad (b-c) = (q-r)d \quad \ldots \text{(i)}, \qquad (c-a) = (r-p)d \quad \ldots \text{(ii)},$$

and $(a-b) = (p-q)d \quad \ldots \text{(iii)}.$

Now, let A be the first term and R be the common ratio of the GP. Then,

$a = AR^{(p-1)}$ … (iv), $b = AR^{(q-1)}$ … (v),

and $c = AR^{(r-1)}$ … (vi).

$$\therefore \quad a^{(b-c)} \cdot b^{(c-a)} \cdot c^{(a-b)}$$
$$= \{AR^{(p-1)}\}^{(q-r)d} \times \{AR^{(q-1)}\}^{(r-p)d} \times \{AR^{(r-1)}\}^{(p-q)d}$$
$$= A^{\{(q-r)d+(r-p)d+(p-q)d\}} \times R^{\{(p-1)(q-r)d+(q-1)(r-p)d+(r-1)(p-q)d\}}$$
$$= A^0 \times R^0 = (1 \times 1) = 1$$

$$\left[\because \quad (q-r)d + (r-p)d + (p-q)d = 0 \right.$$
$$\left. \text{and } p(q-r)d + q(r-p)d + r(p-q)d - (q-r)d - (r-p)d - (p-q)d = 0\right].$$

Hence, $a^{(b-c)} \cdot b^{(c-a)} \cdot c^{(a-b)} = 1$.

EXAMPLE 15 *If $x, 2y, 3z$ are in AP, where the distinct numbers x, y, z are in GP then find the common ratio of the GP.*

SOLUTION It is given that the distinct numbers x, y, z are in GP.

Let r be the common ratio of this GP. Then,

$y = xr$ and $z = xr^2$.

Now, $x, 2y, 3z$ are in AP

$\Rightarrow \quad x, 2xr, 3xr^2$ are in AP

$\Rightarrow \quad 2xr - x = 3xr^2 - 2xr$

$\Rightarrow \quad 3xr^2 - 4xr + x = 0$

$\Rightarrow \quad 3r^2 - 4r + 1 = 0$ $[\because\ x \neq 0$, as it is a term of GP$]$

$\Rightarrow \quad (r-1)(3r-1) = 0 \ \Rightarrow \ r = \frac{1}{3}$ $[\because\ x \neq y \neq z \ \Rightarrow \ r \neq 1]$.

Hence, the common ratio of the GP is $\frac{1}{3}$.

EXAMPLE 16 *In a GP of positive terms, if any term is equal to the sum of the next two terms then show that the common ratio of the GP is $2\sin 18^\circ$.*

SOLUTION Let us consider a GP of positive terms with first term $= a$ and common ratio $= r$. Then, $a > 0$ and $r > 0$.

It is given that $T_n = T_{n+1} + T_{n+2}$.

Now, $T_n = T_{n+1} + T_{n+2}$

$\Rightarrow \quad ar^{n-1} = ar^n + ar^{n+1}$

$\Rightarrow \quad 1 = r + r^2$ [dividing both sides by ar^{n-1}]

$\Rightarrow \quad r^2 + r - 1 = 0$

$\Rightarrow \quad r = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad r = \frac{-1+\sqrt{5}}{2} = 2\left(\frac{\sqrt{5}-1}{4}\right) = 2\sin 18^\circ$ $[\because\ r > 0]$.

Hence, the common ratio of the GP is $2\sin 18^\circ$.

EXAMPLE 17 *If p, q, r are in AP, prove that the pth, qth and rth terms of any GP are in GP.*

SOLUTION Since *p, q, r* are in AP, we have $(q-p)=(r-q)$. … (i)

Let us consider a GP with first term = A and common ratio = R.

Then, $T_p = AR^{(p-1)}$, $T_q = AR^{(q-1)}$ and $T_r = AR^{(r-1)}$.

$$\therefore \quad \frac{T_q}{T_p} = \frac{AR^{(q-1)}}{AR^{(p-1)}} = R^{(q-p)} = R^{(r-q)} \qquad \text{[using (i)]}$$

and $\dfrac{T_r}{T_q} = \dfrac{AR^{(r-1)}}{AR^{(q-1)}} = R^{(r-q)}$.

Thus, $\dfrac{T_r}{T_p} = \dfrac{T_r}{T_q}$ and therefore, T_p, T_q, T_r are in GP.

Hence, the *p*th, *q*th and *r*th terms of any GP are in GP.

EXAMPLE 18 *Let S be the sum, P be the product and R be the sum of the reciprocals of three terms of a GP. Then, prove that* $P^2R^3 = S^3$.

SOLUTION Let the three terms of the given GP be $\dfrac{a}{r}$, a and ar. Then,

$$S = \left(\frac{a}{r} + a + ar\right) = a\left(\frac{1+r+r^2}{r}\right), \; P = \left(\frac{a}{r} \times a \times ar\right) = a^3$$

and $R = \left(\dfrac{r}{a} + \dfrac{1}{a} + \dfrac{1}{ar}\right) = \dfrac{1}{a}\left(r + 1 + \dfrac{1}{r}\right) = \dfrac{1}{a}\left(\dfrac{r^2+r+1}{r}\right) = \dfrac{1}{a}\left(\dfrac{1+r+r^2}{r}\right)$.

$$\therefore \quad P^2R^3 = a^6 \times \frac{(1+r+r^2)^3}{a^3r^3} = a^3\left(\frac{1+r+r^2}{r}\right)^3 = S^3.$$

Hence, $P^2R^3 = S^3$.

EXAMPLE 19 *For a, b, c to be in GP, what is the value of* $\dfrac{a-b}{b-c}$?

SOLUTION For *a, b, c* to be in GP, we must have

$$\frac{b}{a} = \frac{c}{b} = r \quad \text{or} \quad \frac{b}{a} = \frac{c}{b} = \frac{1}{r}.$$

Case I *When* $\dfrac{b}{a} = \dfrac{c}{b} = r$.

In this case, $b = ar$ and $c = br$.

$$\therefore \quad \frac{a-b}{b-c} = \frac{a-ar}{b-br} = \frac{a(1-r)}{b(1-r)} = \frac{a}{b}.$$

Case II *When* $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{1}{r}$.

In this case, $a = br$ and $b = cr$.

$$\therefore \quad \frac{a-b}{b-c} = \frac{br-b}{cr-c} = \frac{b(r-1)}{c(r-1)} = \frac{b}{c}.$$

Hence, the value of $\left(\dfrac{a-b}{b-c}\right)$ is $\dfrac{a}{b}$ or $\dfrac{b}{c}$.

EXAMPLE 20 *If a, b, c are in GP and* $a^{1/x} = b^{1/y} = c^{1/z}$, *prove that x, y, z are in AP.*

SOLUTION Since *a, b, c* are in GP, we have

$$b^2 = ac. \qquad \ldots (i)$$

Let $a^{1/x} = b^{1/y} = c^{1/z} = k$ (say).

Then, $a = k^x$, $b = k^y$ and $c = k^z$.

Putting these values in (i), we get

$$(k^y)^2 = (k^x) \times (k^z) \Rightarrow k^{2y} = k^{(x+z)}.$$

$$\therefore \quad 2y = x + z.$$

Hence, *x, y, z* are in AP.

EXAMPLE 21 *If a, b, c are in AP; b, c, d are in GP and* $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ *are in AP, then show that a, c, e are in GP.*

SOLUTION *a, b, c* are in AP $\Rightarrow 2b = a + c.$ $\ldots$ (i)

b, c, d are in GP $\Rightarrow c^2 = bd.$ $\ldots$ (ii)

$$\frac{1}{c}, \frac{1}{d}, \frac{1}{e} \text{ are in AP} \Rightarrow \frac{2}{d} = \left(\frac{1}{c} + \frac{1}{e}\right) = \frac{(c+e)}{ce}$$

$$\Rightarrow d = \frac{2ce}{(c+e)}. \qquad \ldots (iii)$$

$$\text{Now, } c^2 = bd \Rightarrow c^2 = \frac{(a+c)}{2} \cdot \frac{2ce}{(c+e)} \qquad \text{[from (i) and (iii)]}$$

$$\Rightarrow c = \frac{(a+c)e}{(c+e)}$$

$$\Rightarrow c(c+e) = (a+c)e$$

$$\Rightarrow c^2 = ae$$

$$\Rightarrow a, c, e \text{ are in GP.}$$

Hence, *a, c, e* are in GP.

EXAMPLE 22 *If p, q, r are in GP and the equations*

$$px^2 + 2qx + r = 0 \text{ and } dx^2 + 2ex + f = 0$$

have a common root then show that $\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$ *are in AP.*

SOLUTION Since *p, q, r* are in GP, we have

$$q^2 = pr. \qquad \ldots (i)$$

On solving $px^2 + 2qx + r = 0$, we get

$$x = \frac{-2q \pm \sqrt{4q^2 - 4pr}}{2p} = \frac{-2q}{2p} = \frac{-q}{p} \qquad \text{[using (i)].}$$

Thus, $x = \frac{-q}{p}$ is a repeated root of $px^2 + 2qr + r = 0.$

$$\therefore \quad x = \frac{-q}{p} \text{ is also a root of } dx^2 + 2ex + f = 0$$

$$\Rightarrow \quad d \cdot \left(\frac{-q}{p}\right)^2 + 2e\left(\frac{-q}{p}\right) + f = 0$$

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$$\Rightarrow \quad dq^2 - 2eqp + fp^2 = 0 \qquad \ldots \text{(ii)}$$

$$\Rightarrow \quad \frac{d}{p} - \frac{2e}{q} + \frac{fp}{q^2} = 0 \qquad [\text{on dividing (ii) by } pq^2]$$

$$\Rightarrow \quad \frac{d}{p} - \frac{2e}{q} + \frac{f}{r} = 0 \qquad [\because\ q^2 = pr]$$

$$\Rightarrow \quad \frac{d}{p} + \frac{f}{r} = \frac{2e}{q}.$$

Hence, $\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$ are in AP.

EXAMPLE 23 *Find all sequences which are simultaneously AP and GP.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a sequence which is both an AP as well as a GP.

Consider three consecutive terms a_n, a_{n+1}, a_{n+2} of this AP and GP.

Then, $2a_{n+1} = a_n + a_{n+2}$ ... (i) [$\because$ a_n, a_{n+1}, a_{n+2} are in AP].

And, if r be the common ratio of the GP, then

$$a_n = a_1 r^{n-1},\ a_{n+1} = a_1 r^n \text{ and } a_{n+2} = a_1 r^{n+1}.$$

Putting these values in (i), we get

$$2a_1 r^n = a_1 r^{n-1} + a_1 r^{n+1} \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1.$$

Putting $r = 1$, we get $a_n = a_1, a_{n+1} = a_1$ and $a_{n+2} = a_1$.

Thus, we get a constant sequence $a_1, a_1, a_1, a_1, \ldots$.

Hence, a constant sequence is the only sequence which is both AP and GP.

EXAMPLE 24 *If the pth, qth, rth and sth terms of an AP be in GP then prove that* $(p-q), (q-r), (r-s)$ *are in GP.*

SOLUTION Let a be the first term and d be the common difference of the given AP. Then,

$$T_p = a + (p-1)d,\ T_q = a + (q-1)d,$$

$$T_r = a + (r-1)d,\ T_s = a + (s-1)d.$$

It is given that T_p, T_q, T_r and T_s are in GP.

Let the first term of this GP be A and its common ratio be R. Then,

$$T_p = A \Rightarrow a + (p-1)d = A \qquad \ldots \text{(i)}$$

$$T_q = AR \Rightarrow a + (q-1)d = AR \qquad \ldots \text{(ii)}$$

$$T_r = AR^2 \Rightarrow a + (r-1)d = AR^2 \qquad \ldots \text{(iii)}$$

and $T_s = AR^3 \Rightarrow a + (s-1)d = AR^3.$ ... (iv)

On subtracting (ii) from (i), we get

$$(p-q)d = A(1-R). \qquad \ldots \text{(v)}$$

On subtracting (iii) from (ii), we get

$$(q-r)d = AR(1-R). \qquad \ldots \text{(vi)}$$

On subtracting (iv) from (iii), we get

$$(r-s)d = AR^2(1-R). \qquad \ldots \text{(vii)}$$

Now, we have

$$(q-r)^2d^2 = A^2R^2(1-R)^2 \text{ and } (p-q)d \times (r-s)d = A^2R^2(1-R)^2.$$

$\therefore \quad (q-r)^2d^2 = (p-q)d \times (r-s)d$

$\Rightarrow \quad (q-r)^2 = (p-q) \times (r-s)$

$\Rightarrow \quad (p-q), (q-r)$ and $(r-s)$ are in GP.

Hence, $(p-q), (q-r), (r-s)$ are in GP.

*n*th TERM FROM THE END OF A GP

THEOREM 2 *Show that the nth term from the end of a GP with the first term a, the common ratio r and the last term l is given by* $\frac{l}{r^{(n-1)}}$.

PROOF Let *a* be the first term, *r* be the common ratio and *l* be the last term of a given GP. Then,

$$\text{2nd term from the end} = \frac{l}{r} = \frac{l}{r^{(2-1)}};$$

$$\text{3rd term from the end} = \frac{l}{r^2} = \frac{l}{r^{(3-1)}};$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$n\text{th term from the end} = \frac{l}{r^{(n-1)}}.$$

Hence, the *n*th term from the end $= \frac{l}{r^{(n-1)}}$.

EXAMPLE 25 *Find the 8th term from the end of the GP* $3, 6, 12, 24, \ldots, 12288$.

SOLUTION Here $r = 2$ and $l = 12288$.

$$\therefore \quad \text{8th term from the end} = \frac{l}{r^{(8-1)}} = \frac{l}{r^7} = \frac{12288}{2^7} = \frac{12288}{128} = 96.$$

Hence, the 8th term from the end = 96.

EXERCISE 12A

1. Find the 6th and *n*th terms of the GP $2, 6, 18, 54, \ldots$.
2. Find the 17th and *n*th terms of the GP $2, 2\sqrt{2}, 4, 8\sqrt{2}, \ldots$.
3. Find the 7th and *n*th terms of the GP $0.4, 0.8, 1.6, \ldots$.
4. Find the 10th and *n*th terms of the GP $\frac{-3}{4}, \frac{1}{2}, \frac{-1}{3}, \frac{2}{9}, \ldots$.
5. Which term of the GP $3, 6, 12, 24, \ldots$ is 3072?
6. Which term of the GP $\frac{1}{4}, \frac{-1}{2}, 1, \ldots$ is –128?
7. Which term of the GP $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729?

8. Find the geometric series whose 5th and 8th terms are 80 and 640 respectively.

9. Find the GP whose 4th and 7th terms are $\frac{1}{18}$ and $\frac{-1}{486}$ respectively.

10. The 5th, 8th and 11th terms of a GP are *a*, *b*, *c* respectively. Show that $b^2 = ac$.

11. The first term of a GP is –3 and the square of the second term is equal to its 4th term. Find its 7th term.

12. Find the 6th term from the end of the GP $8, 4, 2, \ldots, \frac{1}{1024}$.

13. Find the 4th term from the end of the GP $\frac{2}{27}, \frac{2}{9}, \frac{2}{3}, \ldots, 162$.

14. If *a*, *b*, *c* are the *p*th, *q*th and *r*th terms of a GP, show that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$.

15. The third term of a GP is 4. Find the product of its first five terms.

16. In a finite GP, prove that the product of the terms equidistant from the beginning and end is the product of first and last terms.

17. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}\ (x \neq 0)$ then show that *a*, *b*, *c*, *d* are in GP.

18. If *a* and *b* are the roots of $x^2 - 3x + p = 0$ and *c* and *d* are the roots of $x^2 - 12x + q = 0$, where *a*, *b*, *c*, *d* form a GP, prove that $(q+p):(q-p) = 17:15$.

ANSWERS (EXERCISE 12A)

1. $486, 2\times 3^{(n-1)}$ **2.** $512, (\sqrt{2})^{(n+1)}$ **3.** $25.6, \frac{2^n}{5}$ **4.** $\frac{128}{6561}, \frac{-3}{4}\times\left(\frac{-2}{3}\right)^{(n-1)}$

5. 11th **6.** 10th **7.** 12th **8.** $5+10+20+40+80+\ldots$

9. $\frac{-3}{2}, \frac{1}{2}, \frac{-1}{6}, \frac{1}{18}, \ldots$ **11.** –2187 **12.** $\frac{1}{32}$ **13.** 6

15. 1024

HINTS TO SOME SELECTED QUESTIONS

14. Let *A* be the first term and *R* be the common ratio of the given GP. Then,

$$a = AR^{(p-1)} \Rightarrow \log a = \log A + (p-1)\log R \quad \ldots \text{(i)}$$

$$b = AR^{(q-1)} \Rightarrow \log b = \log A + (q-1)\log R \quad \ldots \text{(ii)}$$

$$c = AR^{(r-1)} \Rightarrow \log c = \log A + (r-1)\log R. \quad \ldots \text{(iii)}$$

$$\therefore \quad (q-r)\log a + (r-p)\log b + (p-q)\log c$$

$$= (q-r)\log [AR^{(p-1)}] + (r-p)\log \{AR^{(q-1)}] + (p-q)\log [AR^{(r-1)}]$$

$$= (\log A)\{(q-r)+(r-p)+(p-q)\} + (\log R)\{(p-1)(q-r)+(q-1)(r-p) + (r-1)(p-q)\}$$

$$= (\log A)\times 0 + (\log R)\times 0 = 0.$$

15. Let a be the first term and r be the common ratio of the GP. Then, the first five terms are a, ar, ar^2, ar^3, ar^4.

Also, $T_3 = 4 \Rightarrow ar^2 = 4$.

Product of first five terms $= (a \times ar \times ar^2 \times ar^3 \times ar^4) = a^5 r^{10} = (ar^2)^5 = (4)^5 = 1024$.

16. Let a be the first term, r be the common ratio and l be the last term of a GP containing n terms. Then,

(pth term from the beginning) $\times$ (pth term from the end)

$= (p\text{th term from the beginning}) \times \{(n-p+1)\text{th term from the beginning}\}$

$= T_p \times T_{(n-p+1)} = ar^{p-1} \times ar^{(n-p+1)-1} = ar^{(p-1)} \times ar^{(n-p)}$

$= a \times (ar^{n-1}) = (T_1 \times T_n) = (\text{first term} \times \text{last term})$.

17. $\dfrac{a+bx}{a-bx} = \dfrac{b+cx}{b-cx} \Rightarrow \dfrac{(a+bx)+(a-bx)}{(a+bx)-(a-bx)} = \dfrac{(b+cx)+(b-cx)}{(b+cx)-(b-cx)} \Rightarrow \dfrac{b}{a} = \dfrac{c}{b}$.

Similarly, $\dfrac{b+cx}{b-cx} = \dfrac{c+dx}{c-dx} \Rightarrow \dfrac{c}{b} = \dfrac{d}{c}$.

$\therefore \quad \dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} \Rightarrow a, b, c, d$ are in GP.

18. Let r be the common ratio of the given GP.

Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

Also, $a + b = 3$, $ab = p$, $c + d = 12$ and $cd = q$.

Now, $a + b = 3, c + d = 12 \Rightarrow a(1+r) = 3$ and $ar^2(1+r) = 12$

$$\Rightarrow \frac{ar^2(1+r)}{a(1+r)} = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$$

$$\Rightarrow a(1+2) = 3 \Rightarrow a = 1.$$

$\therefore \quad p = ab = a \times ar = a^2 r = (1^2 \times 2) = 2$ and $q = cd = ar^2 \times ar^3 = a^2 r^5 = (1)^2 \times 2^5 = 32$.

Hence, $\dfrac{q+p}{q-p} = \dfrac{32+2}{32-2} = \dfrac{34}{30} = \dfrac{17}{15}$.

PROBLEMS ON GP

For solving problems on GP, it is always convenient to take:

(i) 3 numbers in GP as $\dfrac{a}{r}, a\ ar$;

(ii) 4 numbers in GP $\dfrac{a}{r^3}, \dfrac{a}{r}, ar, ar^3$;

(iii) 5 numbers in GP as $\dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2$;

(iv) the terms as $a, ar, ar^2, \ldots$ when their product is not given.

SOLVED EXAMPLES

EXAMPLE 1 *The sum of first three terms of a GP is* $\dfrac{13}{12}$ *and their product is* -1. *Find these terms.*

SOLUTION Let the first three terms of the given GP be $\frac{a}{r}$, a and ar.

Then, $\frac{a}{r} \times a \times ar = -1 \Rightarrow a^3 = -1 \Rightarrow a = -1.$

And, $\frac{a}{r} + a + ar = \frac{13}{12} \Rightarrow \frac{-1}{r} - 1 - r = \frac{13}{12}$ [$\because a = -1$]

$$\Rightarrow \frac{-1}{r} - r = \frac{13}{12} + 1$$

$$\Rightarrow \frac{-1 - r^2}{r} = \frac{25}{12}$$

$$\Rightarrow 25r = -12 - 12r^2$$

$$\Rightarrow 12r^2 + 25r + 12 = 0$$

$$\Rightarrow 12r^2 + 16r + 9r + 12 = 0$$

$$\Rightarrow 4r(3r + 4) + 3(3r + 4) = 0$$

$$\Rightarrow (3r + 4)(4r + 3) = 0 \Rightarrow r = \frac{-4}{3} \text{ or } r = \frac{-3}{4}.$$

So, the required terms are

$$\left\{\frac{-1}{(-4/3)}, -1, \frac{4}{3}\right\} \text{ or } \left\{\frac{-1}{(-3/4)}, -1, \frac{3}{4}\right\}$$

i.e., $\frac{3}{4}, -1, \frac{4}{3}$ or $\frac{4}{3}, -1, \frac{3}{4}$.

EXAMPLE 2 *Find three numbers in GP whose sum is* 13 *and the sum of whose squares is* 91.

SOLUTION Let the required numbers be $\frac{a}{r}$, a and ar. Then,

$$\frac{a}{r} + a + ar = 13 \quad \ldots \text{(i)}$$

and $$\frac{a^2}{r^2} + a^2 + a^2r^2 = 91. \quad \ldots \text{(ii)}$$

On squaring both sides of (i), we get

$$\left(\frac{a}{r} + a + ar\right)^2 = 169$$

$$\Rightarrow \left(\frac{a^2}{r^2} + a^2 + a^2r^2\right) + 2\left(\frac{a^2}{r} + a^2 + a^2r\right) = 169$$

$$\Rightarrow 91 + 2a\left(\frac{a}{r} + a + ar\right) = 169 \quad \text{[using (ii)]}$$

$$\Rightarrow 91 + 26a = 169 \quad \text{[using (i)]}$$

$$\Rightarrow 26a = 78 \Rightarrow a = 3.$$

Putting $a = 3$ in (i), we get

$$\frac{3}{r} + 3 + 3r = 13 \Rightarrow \frac{3}{r} + 3r = 10 \Rightarrow 3r^2 - 10r + 3 = 0$$

$$\Rightarrow (r - 3)(3r - 1) = 0 \Rightarrow r = 3 \text{ or } r = \frac{1}{3}.$$

Hence, the required numbers are 1, 3, 9 or 9, 3, 1.

EXAMPLE 3 *Find three numbers in GP whose sum is 52 and the sum of whose products in pairs is 624.*

SOLUTION Let the required numbers be a, ar, ar^2. Then,

$$(a + ar + ar^2) = 52 \Rightarrow a(1 + r + r^2) = 52 \quad \ldots \text{(i)}$$

and $a \cdot ar + ar \cdot ar^2 + a \cdot ar^2 = 624 \Rightarrow a^2 r(1 + r + r^2) = 624. \quad \ldots \text{(ii)}$

On dividing (ii) by (i), we get

$ar = 12$ and therefore, $a = \frac{12}{r}$.

Putting $a = \frac{12}{r}$ in (i), we get

$$\frac{12}{r} \cdot (1 + r + r^2) = 52 \Rightarrow 3(1 + r + r^2) = 13r$$

$$\Rightarrow 3r^2 - 10r + 3 = 0$$

$$\Rightarrow (3r - 1)(r - 3) = 0$$

$$\Rightarrow r = \frac{1}{3} \text{ or } r = 3.$$

$\therefore \quad a = 36$ or $a = 4$.

Hence, the required numbers are 36, 12, 4 or 4, 12, 36.

EXAMPLE 4 *If the product of three numbers in GP is 216 and the sum of their products in pairs is 156, find the numbers.*

SOLUTION Let the required numbers be $\frac{a}{r}$, a and ar. Then,

$$\frac{a}{r} \times a \times ar = 216 \Rightarrow a^3 = 216 = 6^3 \Rightarrow a = 6.$$

And, $\frac{a}{r} \times a + a \times ar + \frac{a}{r} \times ar = 156$

$$\Rightarrow \quad a^2\left(\frac{1}{r} + r + 1\right) = 156 \Rightarrow (6^2)(1 + r + r^2) = 156r \qquad [\because a = 6]$$

$$\Rightarrow \quad 36(r^2 + r + 1) = 156r \Rightarrow 3(r^2 + r + 1) = 13r$$

$$\Rightarrow \quad 3r^2 - 10r + 3 = 0 \Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = \frac{1}{3} \text{ or } r = 3.$$

So, the required numbers are 18, 6, 2 or 2, 6, 18.

EXAMPLE 5 *Find four numbers in GP such that the third term is greater than the first by 9 and the second term is greater than the fourth by 18.*

SOLUTION Let the required numbers be a, ar, ar^2 and ar^3. Then,

$$T_3 - T_1 = 9 \Rightarrow ar^2 - a = 9 \Rightarrow a(r^2 - 1) = 9 \quad \ldots \text{(i)}$$

and $T_2 - T_4 = 18 \Rightarrow ar - ar^3 = 18 \Rightarrow ar(1 - r^2) = 18. \quad \ldots \text{(ii)}$

On dividing (ii) by (i), we get $r = -2$.

Putting $r = -2$ in (i), we get $a(4 - 1) = 9 \Rightarrow a = 3$.

Hence, the required numbers are 3, –6, 12 and –24.

EXAMPLE 6 *The sum of three numbers in GP is 56. If we subtract 1, 7, 21 from these numbers in that order, we get an AP. Find the numbers.*

SOLUTION Let the required numbers be a, ar and ar^2. Then,

$$a + ar + ar^2 = 56. \qquad \ldots (i)$$

Also, $(a-1)$, $(ar-7)$ and (ar^2-21) are in AP.

$$\therefore \quad 2(ar-7) = (a-1) + (ar^2-21) \Rightarrow a + ar^2 = 2ar + 8. \qquad \ldots (ii)$$

Using (ii) in (i), we get

$$ar + (2ar+8) = 56 \Rightarrow 3ar = 48 \Rightarrow ar = 16 \Rightarrow r = \frac{16}{a}.$$

Putting $r = \frac{16}{a}$ in (i), we get

$$a + 16 + \frac{256}{a} = 56 \Rightarrow a^2 + 16a + 256 = 56a$$
$$\Rightarrow a^2 - 40a + 256 = 0$$
$$\Rightarrow a^2 - 8a - 32a + 256 = 0$$
$$\Rightarrow a(a-8) - 32(a-8) = 0$$
$$\Rightarrow (a-8)(a-32) = 0$$
$$\Rightarrow a = 8 \text{ or } a = 32.$$

Now, $a = 8 \Rightarrow r = \frac{16}{8} \Rightarrow r = 2.$

And, $a = 32 \Rightarrow r = \frac{16}{32} \Rightarrow r = \frac{1}{2}.$

Hence, the required numbers are (8, 16, 32) or (32, 16, 8).

EXERCISE 12B

1. For what values of x are the numbers $\frac{-2}{7}$, x, $\frac{-7}{2}$ in GP?
2. For what values of x are the numbers $(x+9)$, $(x-6)$ and 4 in GP?
3. The sum of three numbers in GP is $\frac{39}{10}$ and their product is 1. Find the numbers.
4. The sum of first three terms of a GP is $\frac{13}{12}$ and their product is –1. Find the GP.
5. Find three numbers in GP whose sum is 38 and whose product is 1728.
6. Find three numbers in GP whose sum is 65 and whose product is 3375.
7. The sum of three numbers in GP is 21 and the sum of their squares is 189. Find the numbers.
8. The product of three numbers in GP is 216. If 2, 8, 6 be added to them in that order, we get an AP. Find the numbers.
9. The product of three numbers in GP is 1000. If 6 is added to the second number and 7 is added to the third number, we get an AP. Find the numbers.

1. $x = 1$ or $x = -1$ **2.** $x = 0$ or $x = 16$ **3.** $\frac{5}{2}, 1, \frac{2}{5}$ or $\frac{2}{5}, 1, \frac{5}{2}$

4. $\frac{4}{3}, -1, \frac{3}{4}$ or $\frac{3}{4}, -1, \frac{4}{3}$ **5.** 18, 12, 8 or 8, 12, 18 **6.** 45, 15, 5 or 5, 15, 45

7. 12, 6, 3 or 3, 6, 12 **8.** 18, 6, 2 or 2, 6, 18 **9.** 5, 10, 20 or 20, 10, 5

SUM OF *n* TERMS OF A GP

THEOREM *Prove that the sum of n terms of a GP with the first term a and the common ratio r is given by*

$$S_n = \begin{cases} na, \text{ when } r = 1; \\ \dfrac{a(1-r^n)}{(1-r)}, \text{ when } r < 1; \\ \dfrac{a(r^n-1)}{(r-1)}, \text{ when } r > 1. \end{cases}$$

PROOF Let us consider a GP with the first term a and the common ratio r. Then,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}. \qquad \ldots \text{(i)}$$

Case 1 If $r = 1$, we have

$$S_n = a + a + \ldots \text{ to } n \text{ terms} = na.$$

Case 2 If $r \neq 1$, we have

$$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n. \qquad \ldots \text{(ii)}$$

On subtracting (ii) from (i), we get

$$(1-r)S_n = (a - ar^n) = a(1-r^n)$$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{(1-r)} \text{ or } S_n = \frac{a(r^n-1)}{(r-1)}.$$

$$\therefore \quad S_n = \begin{cases} \dfrac{a(1-r^n)}{(1-r)}, \text{ when } r < 1 \\ \dfrac{a(r^n-1)}{(r-1)}, \text{ when } r > 1 \end{cases}$$

REMARK If a GP contains n terms with first term $= a$, common ratio $= r$ and last term $= l$ then

$$l = ar^{n-1}. \qquad \ldots \text{(i)}$$

Case 1 When $r < 1$, we have

$$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{(a - ar^n)}{(1-r)} = \frac{(a-lr)}{(1-r)} \qquad \text{[using (i)].}$$

Case 2 When $r > 1$, we have

$$S_n = \frac{a(r^n - 1)}{(r-1)} = \frac{(ar^n - a)}{(r-1)} = \frac{(lr - a)}{(r-1)} \qquad \text{[using (i)].}$$

SUMMARY

In a GP with first term = a and common ratio = r, we have

$$S_n = \begin{cases} na, \text{ when } r = 1; \\ \dfrac{a(1-r^n)}{(1-r)} = \dfrac{(a-lr)}{(1-r)}, \text{ when } r < 1; \\ \dfrac{a(r^n-1)}{(r-1)} = \dfrac{(lr-a)}{(r-1)}, \text{ when } r > 1. \end{cases}$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the sum of* 8 *terms of the GP* 3, 6, 12, 24, … .

SOLUTION Here $a = 3$, $r = 2 > 1$ and $n = 8$.

Using the formula, $S_n = \dfrac{a(r^n - 1)}{(r-1)}$, we get

$$S_8 = \frac{3 \times (2^8 - 1)}{(2-1)} = 3 \times (256 - 1) = 3 \times 255 = 765.$$

EXAMPLE 2 *Find the sum of the geometric series* $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ *to* 12 *terms.*

SOLUTION Here $a = 1$, $r = \dfrac{1}{2} < 1$ and $n = 12$.

Using the formula, $S_n = \dfrac{a(1-r^n)}{(1-r)}$, we get

$$S_{12} = \frac{1 \times \left\{1 - \left(\frac{1}{2}\right)^n\right\}}{\left(1 - \frac{1}{2}\right)} = \frac{\left(1 - \frac{1}{2^{12}}\right)}{\left(\frac{1}{2}\right)} = \frac{(2^{12} - 1)}{2^{11}} = \frac{4095}{2048}.$$

EXAMPLE 3 *How many terms of the geometric series* $1 + 4 + 16 + 64 + \ldots$ *will make the sum* 5461*?*

SOLUTION Let the required number of terms be *n*.

Here, $a = 1$, $r = 4 > 1$ and $S_n = 5461$.

$$\therefore \quad S_n = \frac{a(r^n - 1)}{(r-1)} \Rightarrow \frac{1 \times (4^n - 1)}{(4-1)} = 5461$$

$$\Rightarrow \quad (4^n - 1) = 16383$$

$$\Rightarrow \quad 4^n = 16384 = 4^7$$

$$\Rightarrow \quad n = 7.$$

Hence, the required number of terms is 7.

EXAMPLE 4 *Find the sum of the series* $2+6+18+54+\ldots+4374$.

SOLUTION Clearly, the given series is a geometric series in which $a=2$, $r=3>1$ and $l=4374$.

$$\therefore \quad \text{the required sum} = \frac{(lr-a)}{(r-1)} = \frac{(4374\times 3-2)}{(3-1)} = \frac{13120}{2} = 6560.$$

Hence, the sum of the given series is 6560.

EAMPLE 5 *In a GP, it is being given that* $T_1=3$, $T_n=96$ *and* $S_n=189$. *Find the value of n.*

SOLUTION Here, $a=3$, $l=96$ and $S_n=189$.

Let the common ratio of the given GP be *r*.

$$\text{Then, } S_n=\frac{(lr-a)}{(r-1)} \Rightarrow \frac{(96r-3)}{(r-1)}=189$$

$$\Rightarrow (96r-3)=(189r-189)$$

$$\Rightarrow 93r=186 \Rightarrow r=2.$$

$$\text{Now, } l=ar^{n-1} \Rightarrow 3\times 2^{n-1}=96$$

$$\Rightarrow 2^{n-1}=32=2^5 \Rightarrow n-1=5 \Rightarrow n=6.$$

Hence, $n=6$.

EXAMPLE 6 *Sum the series* $5+55+555+\ldots$ *to n terms.*

SOLUTION We have

$$5+55+555+\ldots \text{ to } n \text{ terms}$$

$$=5\times\{1+11+111+\ldots \text{ to } n \text{ terms}\}$$

$$=\frac{5}{9}\times\{9+99+999+\ldots \text{ to } n \text{ terms}\}$$

$$=\frac{5}{9}\times\{(10-1)+(10^2-1)+(10^3-1)+\ldots \text{ to } n \text{ terms}\}$$

$$=\frac{5}{9}\times\{(10+10^2+10^3+\ldots \text{ to } n \text{ terms})-n\}$$

$$=\frac{5}{9}\times\left\{\frac{10\times(10^n-1)}{(10-1)}-n\right\}=\frac{5}{81}\times(10^{n+1}-9n-10).$$

Hence, the required sum is $\frac{5}{81}\times(10^{n+1}-9n-10)$.

EXAMPLE 7 *Sum the series* $.4+.44+.444+\ldots$ *to n terms.*

SOLUTION We have

$$.4+.44+.444+\ldots \text{ to } n \text{ terms}$$

$$=4\times\{.1+.11+.111+\ldots \text{ to } n \text{ terms}\}$$

$$=\frac{4}{9}\times\{.9+.99+.999+\ldots \text{ to } n \text{ terms}\}$$

$$=\frac{4}{9}\times\{(1-.1)+(1-.01)+(1-.001)+\ldots \text{ to } n \text{ terms}\}$$

$$= \frac{4}{9} \times \{(1+1+\ldots \text{ to } n \text{ terms}) - (.1+.01+.001+\ldots \text{ to } n \text{ terms})\}$$

$$= \frac{4}{9} \times \left[n - \frac{.1 \times \{1-(.1)^n\}}{(1-.1)} \right] \qquad \left\{ \because\ S_n = \frac{a(1-r^n)}{(1-r)} \right\}$$

$$= \frac{4}{9} \times \left[n - \frac{\frac{1}{10} \cdot \left\{1 - \frac{1}{(10)^n}\right\}}{\left(1-\frac{1}{10}\right)} \right]$$

$$= \frac{4}{9} \times \left\{ n - \frac{(10^n - 1)}{9 \cdot 10^n} \right\} = \frac{4}{9} \times \left[n - \frac{1}{9}\left\{1 - \frac{1}{10^n}\right\} \right]$$

$$= \frac{4}{81} \times \left[9n - \left\{1 - \frac{1}{10^n}\right\} \right] = \frac{4}{81} \times \left\{ 9n - 1 + \frac{1}{10^n} \right\}.$$

Hence, the required sum is $\frac{4}{81} \times \left(9n - 1 + \frac{1}{10^n}\right)$.

EXAMPLE 8 *The sum of some terms of a GP is* 315. *Its first term is* 5 *and the common ratio is* 2. *Find the number of its terms and the last term.*

SOLUTION Let the given GP contain n terms. Then,

$$a = 5,\ r = 2 > 1 \text{ and } S_n = 315.$$

$$\therefore \quad S_n = 315 \ \Rightarrow\ \frac{a(r^n - 1)}{(r-1)} = 315$$

$$\Rightarrow\ \frac{5 \times (2^n - 1)}{(2-1)} = 315$$

$$\Rightarrow\ (2^n - 1) = 63 \ \Rightarrow\ 2^n = 64 = 2^6 \ \Rightarrow\ n = 6.$$

Last term $= ar^{(n-1)} = 5 \times 2^{(6-1)} = (5 \times 2^5) = (5 \times 32) = 160.$

Hence, the given GP contains 6 terms and its last term is 160.

EXAMPLE 9 *The sum of first three terms of a GP is* 16 *and the sum of its next three terms is* 128. *Find the sum of n terms of the GP.*

SOLUTION Let a be the first term and r be the common ratio of the given GP.

Then, $a + ar + ar^2 = 16$ and $ar^3 + ar^4 + ar^5 = 128$

$$\Rightarrow \quad a(1 + r + r^2) = 16 \quad \ldots \text{(i)} \quad \text{and} \quad ar^3(1 + r + r^2) = 128 \quad \ldots \text{(ii)}.$$

On dividing (ii) by (i), we get $r^3 = 8 = 2^3 \ \Rightarrow\ r = 2.$

Putting $r = 2$ in (i), we get

$$a(1 + 2 + 4) = 16 \ \Rightarrow\ a = \frac{16}{7}.$$

Thus, in the given GP, we have $a = \frac{16}{7}$ and $r = 2 > 1.$

$$\therefore \quad S_n = \frac{a(r^n - 1)}{(r-1)} = \frac{\frac{16}{7} \times (2^n - 1)}{(2-1)} = \frac{16}{7}(2^n - 1).$$

EXAMPLE 10 *In a GP, the sum of first two terms is –4 and the 5th term is 4 times the 3rd term. Find the GP.*

SOLUTION Let a be the first term and r be the common ratio of the given GP. Then,

$$T_1 + T_2 = -4 \text{ and } T_5 = 4 \times T_3$$

$$\Rightarrow \quad a + ar = -4 \text{ and } ar^4 = 4 \times ar^2$$

$$\Rightarrow \quad a(1+r) = -4 \text{ and } r^2 = 4$$

$$\Rightarrow \quad a(1+r) = -4 \quad \ldots \text{(i)} \quad \text{and} \quad r = \pm 2.$$

Putting $r = 2$ in (i), we get $a = \frac{-4}{3}$.

Putting $r = -2$ in (i), we get $a = 4$.

$\therefore \quad a = \frac{-4}{3}$ and $r = 2$ gives the GP $\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, \ldots$

and, $a = 4$ and $r = -2$ gives the GP $4, -8, 16, \ldots$.

Hence, the required GP is $\left\{\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, \ldots\right\}$ or $\{4, -8, 16, \ldots\}$.

EXAMPLE 11 *In an increasing GP, the sum of the first and last terms is 66, the product of the second and the last but one is 128 and the sum of the terms is 126. How many terms are there in this GP?*

SOLUTION Let the given GP contain n terms. Let a be the first term and r be the common ratio of this GP.

Since the given GP is increasing, we have $r > 1$.

Now, $T_1 + T_n = 66 \Rightarrow a + ar^{(n-1)} = 66.$ … (i)

And, $T_2 \times T_{n-1} = 128 \Rightarrow ar \times ar^{(n-2)} = 128$

$$\Rightarrow a^2 r^{(n-1)} = 128 \Rightarrow ar^{(n-1)} = \frac{128}{a}. \quad \ldots \text{(ii)}$$

Using (ii) in (i), we get

$$a + \frac{128}{a} = 66 \Rightarrow a^2 - 66a + 128 = 0$$

$$\Rightarrow a^2 - 2a - 64a + 128 = 0$$

$$\Rightarrow a(a-2) - 64(a-2) = 0$$

$$\Rightarrow (a-2)(a-64) = 0$$

$$\Rightarrow a = 2 \text{ or } a = 64.$$

Putting $a = 2$ in (ii), we get

$$r^{(n-1)} = \frac{128}{a^2} = \frac{128}{4} = 32. \quad \ldots \text{(iii)}$$

Putting $a = 64$ in (ii), we get

$$r^{(n-1)} = \frac{128}{a^2} = \frac{128}{64 \times 64} = \frac{1}{32}, \text{ which is rejected, since } r > 1.$$

Thus, $a = 2$ and $r^{(n-1)} = 32$.

Now, $S_n = 126 \Rightarrow \dfrac{a(r^n-1)}{(r-1)} = 126$

$\Rightarrow 2\left(\dfrac{r^n-1}{r-1}\right) = 126 \Rightarrow \dfrac{r^n-1}{r-1} = 63$

$\Rightarrow \dfrac{r^{(n-1)} \times r - 1}{r-1} = 63 \Rightarrow \dfrac{32r-1}{r-1} = 63$

$\Rightarrow 32r - 1 = 63r - 63 \Rightarrow 31r = 62 \Rightarrow r = 2.$

$\therefore \quad r^{(n-1)} = 32 = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6.$

Hence, there are 6 terms in the given GP.

REMARK Consider two GPs $a, ar, ar^2, ar^3, \ldots$ *and* $A, AR, AR^2, AR^3, \ldots$.

Taking the products of their corresponding elements, we get

$aA, aA(rR), aA(rR)^2, aA(rR)^3, \ldots$

which is clearly a GP with first term = aA and common ratio = rR.

Based upon this result, we take the following example.

EXAMPLE 12 *Find the sum of the products of the corresponding terms of finite geometrical progressions*

$2, 4, 8, 16, 32$ *and* $128, 32, 8, 2, \dfrac{1}{2}$.

SOLUTION Taking the products of the corresponding terms of two given GPs, we get a new progression

$(2 \times 128), (4 \times 32), (8 \times 8), (16 \times 2), \left(32 \times \dfrac{1}{2}\right),$

i.e., 256, 128, 64, 32, 16, which is clearly a GP in which $a = 256$ and $r = \dfrac{16}{32} = \dfrac{1}{2} < 1.$

$\therefore$ the required sum $= \dfrac{a(1-r^5)}{(1-r)} = \dfrac{256 \times \left\{1 - \left(\dfrac{1}{2}\right)^5\right\}}{\left(1 - \dfrac{1}{2}\right)}$

$= \dfrac{256 \times \left(1 - \dfrac{1}{2^5}\right)}{\left(\dfrac{1}{2}\right)} = 256 \times 2 \times \left(1 - \dfrac{1}{32}\right)$

$= \left(256 \times 2 \times \dfrac{31}{32}\right) = 496.$

EXAMPLE 13 *Find the sum of n terms of the sequence given by* $a_n = (3^n + 5n), n \in N,$

SOLUTION Let the sum of *n* terms of the given sequence be S_n. Then,

$S_n = a_1 + a_2 + a_3 + \ldots + a_n$

$= (3^1 + 5 \times 1) + (3^2 + 5 \times 2) + (3^3 + 5 \times 3) + \ldots + (3^n + 5 \times n)$

$= (3 + 3^2 + 3^3 + \ldots + 3^n) + (5 \times 1 + 5 \times 2 + 5 \times 3 + \ldots + 5 \times n)$

$$= (3 + 3^2 + 3^3 + \ldots + 3^n) + 5 \times (1 + 2 + 3 + \ldots + n)$$

$$= \frac{3(3^n - 1)}{(3-1)} + 5 \times \frac{n}{2}(1 + n)$$

$$= \frac{3}{2}(3^n - 1) + \frac{5}{2} \times n(n+1).$$

Hence, the required sum is $\frac{3}{2}(3^n - 1) + \frac{5}{2}n(n+1)$.

EXAMPLE 14 *If S_1, S_2 and S_3 be respectively the sum of n, $2n$ and $3n$ terms of a GP then prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$.*

SOLUTION Let *a* be the first term and *r* be the common ratio of the given GP. Then,

$$S_1(S_3 - S_2) = \frac{a(1-r^n)}{(1-r)} \times \left\{\frac{a(1-r^{3n})}{(1-r)} - \frac{a(1-r^{2n})}{(1-r)}\right\}$$

$$= \frac{a(1-r^n)}{(1-r)} \times \frac{(a - ar^{3n} - a + ar^{2n})}{(1-r)}$$

$$= \frac{a(1-r^n)}{(1-r)} \times \frac{ar^{2n}(1-r^n)}{(1-r)}$$

$$= \frac{a^2 r^{2n}(1-r^n)^2}{(1-r)^2}.$$

And, $(S_2 - S_1)^2 = \left\{\frac{a(1-r^{2n})}{(1-r)} - \frac{a(1-r^n)}{(1-r)}\right\}^2$

$$= \frac{(a - ar^{2n} - a + ar^n)^2}{(1-r)^2}$$

$$= \frac{\{ar^n(1-r^n)\}^2}{(1-r)^2}$$

$$= \frac{a^2 r^{2n}(1-r^n)^2}{(1-r)^2}.$$

Hence, $S_1(S_3 - S_2) = (S_2 - S_1)^2$.

EXAMPLE 15 *If S be the sum, P be the product and R be the sum of the reciprocals of n terms in a GP, prove that $P^2 = \left(\frac{S}{R}\right)^n$.*

SOLUTION Let the given GP be $a, ar, ar^2, \ldots, ar^{(n-1)}$. Then,

$$S = \frac{a(1-r^n)}{(1-r)} \qquad \ldots \text{(i)}$$

and $P = [a \times ar \times ar^2 \times \ldots \times ar^{(n-1)}]$

$$\Rightarrow \quad P = a^n r^{[1+2+3+\ldots+(n-1)]} = a^n r^{\frac{1}{2}(n-1)n}$$

$$\Rightarrow \quad P = a^n r^{\frac{1}{2}(n-1)n}$$

$$\Rightarrow \quad P^2 = a^{2n} r^{(n-1)n}. \qquad \ldots \text{(ii)}$$

And, $R = \left\{\frac{1}{a} + \frac{1}{ar} + \ldots + \frac{1}{ar^{n-1}}\right\} = \left(\frac{1}{a}\right) \cdot \frac{\left\{\frac{1}{r^n} - 1\right\}}{\left\{\frac{1}{r} - 1\right\}}$

$$\Rightarrow \quad R = \left(\frac{r}{a}\right) \cdot \frac{(1 - r^n)}{(1 - r) \cdot r^n}$$

$$\Rightarrow \quad R = \frac{(1 - r^n)}{a(1 - r) \cdot r^{(n-1)}} \cdot \qquad \ldots \text{(iii)}$$

On dividing (i) by (iii), we get

$$\frac{S}{R} = \frac{a(1 - r^n)}{(1 - r)} \cdot \frac{a(1 - r) \cdot r^{(n-1)}}{(1 - r^n)} = a^2 r^{(n-1)}$$

$$\therefore \quad \left(\frac{S}{R}\right)^n = \{a^2 r^{(n-1)}\}^n = a^{2n} \cdot r^{(n-1)n} = P^2 \qquad \text{[using (ii)].}$$

Hence, $P^2 = \left(\frac{S}{R}\right)^n$.

EXERCISE 12C

1. Find the sum of the GP:

(i) $1 + 3 + 9 + 27 + \ldots$ to 7 terms

(ii) $1 + \sqrt{3} + 3 + 3\sqrt{3} + \ldots$ to 10 terms

(iii) $0.15 + 0.015 + 0.0015 + \ldots$ to 6 terms

(iv) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ to 9 terms

(v) $\sqrt{2} + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} + \ldots$ to 8 terms

(vi) $\frac{2}{9} - \frac{1}{3} + \frac{1}{2} - \frac{3}{4} + \ldots$ to 6 terms

2. Find the sum of the GP:

(i) $\sqrt{7} + \sqrt{21} + 3\sqrt{7} + \ldots$ to n terms

(ii) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \ldots$ to n terms

(iii) $1 - a + a^2 - a^3 + \ldots$ to n terms $(a \neq 1)$

(iv) $x^3 + x^5 + x^7 + \ldots$ to n terms

(v) $x(x + y) + x^2(x^2 + y^2) + x^3(x^3 + y^3) + \ldots$ to n terms

3. Find the sum to n terms of the sequence:

(i) $\left(x+\frac{1}{x}\right)^2, \left(x^2+\frac{1}{x^2}\right)^2, \left(x^3+\frac{1}{x^3}\right)^2, \ldots$ to n terms

(ii) $(x+y), (x^2+xy+y^2), (x^3+x^2y+xy^2+y^3), \ldots$ to n terms

4. Find the sum:

$$\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\ldots \text{ to } 2n \text{ terms.}$$

5. Evaluate:

(i) $\sum_{n=1}^{10}(2+3^n)$ (ii) $\sum_{k=1}^{n}[2^k+3^{(k-1)}]$ (iii) $\sum_{n=1}^{8}5^n$

6. Find the sum of the series:

(i) $8+88+888+\ldots$ to n terms

(ii) $3+33+333+\ldots$ to n terms

(iii) $0.7+0.77+0.777+\ldots$ to n terms

7. The sum of n terms of a progression is (2^n-1). Show that it is a GP and find its common ratio.

8. In a GP, the ratio of the sum of first three terms is to that of first six terms is 125 : 152. Find the common ratio.

9. Find the sum of the geometric series $3+6+12+\ldots+1536$.

10. How many terms of the series $2+6+18+\ldots$ must be taken to make the sum equal to 728?

11. The common ratio of a finite GP is 3 and its last term is 486. If the sum of these terms is 728, find the first term.

12. The first term of a GP is 27 and its 8th term is $\frac{1}{81}$. Find the sum of its first 10 terms.

13. The 2nd and 5th terms of a GP are $\frac{-1}{2}$ and $\frac{1}{16}$ respectively. Find the sum of the GP up to 8 terms.

14. The 4th and 7th terms of a GP are $\frac{1}{27}$ and $\frac{1}{729}$ respectively. Find the sum of n terms of the GP.

15. A GP consists of an even number of terms. If the sum of all the terms is 5 times the sum of the terms occupying the odd places, find the common ratio of the GP.

16. Show that the ratio of the sum of first n terms of a GP to the sum of the terms from $(n+1)$th to $(2n)$th term is $\frac{1}{r^n}$.

ANSWERS (EXERCISE 12C)

1. (i) 1093 (ii) $121(\sqrt{3}+1)$ (iii) $\frac{333333}{2000000}$ (iv) $\frac{171}{256}$

(v) $\frac{255\sqrt{2}}{128}$ (vi) $\frac{-133}{144}$

2. (i) $\frac{\sqrt{7}}{2}(\sqrt{3}+1)(3^{n/2}-1)$ (ii) $\frac{\{3^n-(-1)^n\}}{4\times 3^{(n-1)}}$ (iii) $\frac{\{1-(-a)^n\}}{(1+a)}$

(iv) $\frac{x^3\times(x^{2n}-1)}{(x^2-1)}$ (v) $\frac{x^2(x^{2n}-1)}{(x^2-1)}+\frac{xy(x^n y^n-1)}{(xy-1)}$

3. (i) $\left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right)+2n$ (ii) $\frac{1}{(x-y)}\cdot\left\{x^2\left(\frac{x^n-1}{x-1}\right)-y^2\left(\frac{y^n-1}{y-1}\right)\right\}$

4. $\frac{19}{24}\cdot\left(\frac{5^{2n}-1}{5^{2n}}\right)$ **5.** (i) 29544 (ii) $\frac{1}{2}[2^{(n+2)}+3^n-5]$ (iii) 488280

6. (i) $\frac{8}{81}[10^{(n+1)}-10-9n]$ (ii) $\frac{1}{27}[10^{(n+1)}-10-9n]$ (iii) $\frac{7}{81}\left(9n-1+\frac{1}{10^n}\right)$

7. $r=2$ **8.** $r=\frac{3}{5}$ **9.** 3069 **10.** 6 **11.** $a=2$

12. $\frac{81}{2}\left(1-\frac{1}{3^{10}}\right)$ **13.** $\frac{85}{128}$ **14.** $\frac{3}{2}\left(1-\frac{1}{3^n}\right)$ **15.** $r=4$

HINTS TO SOME SELECTED QUESTIONS

1. (i) $S_7=\frac{1\times(3^7-1)}{(3-1)}=\frac{(2187-1)}{2}=\frac{2186}{2}=1093.$

(ii) $S_{10}=\frac{1\times\{(\sqrt{3})^{10}-1\}}{(\sqrt{3}-1)}=\frac{(3^5-1)}{(\sqrt{3}-1)}\times\frac{(\sqrt{3}+1)}{(\sqrt{3}+1)}=\frac{242(\sqrt{3}+1)}{(3-1)}=121(\sqrt{3}+1).$

(iii) $a=\frac{15}{100}=\frac{3}{20}$ and $r=\frac{0.015}{0.15}=\frac{1}{10}$.

$$\therefore\quad S_6=\frac{\frac{3}{20}\times\left(1-\frac{1}{10^6}\right)}{\left(1-\frac{1}{10}\right)}=\left(\frac{3}{20}\times\frac{10}{9}\right)\cdot\frac{(1000000-1)}{1000000}=\frac{1}{6}\times\frac{999999}{1000000}=\frac{333333}{2000000}.$$

(iv) $a=1, r=\frac{-1}{2}$ and $n=9$.

$$\therefore\quad S_9=\frac{1\times\left[1-\left(\frac{-1}{2}\right)^9\right]}{\left(1+\frac{1}{2}\right)}=\frac{2}{3}\times\left(1+\frac{1}{2^9}\right)=\left(\frac{2}{3}\times\frac{513}{512}\right)=\frac{171}{256}.$$

(v) $a=\sqrt{2}, r=\frac{1}{2}$ and $n=8$.

$$\therefore\quad S_8=\frac{\sqrt{2}\left(1-\frac{1}{2^8}\right)}{\left(1-\frac{1}{2}\right)}=\left(\sqrt{2}\times\frac{255}{256}\times 2\right)=\frac{255\sqrt{2}}{128}.$$

(vi) $a=\frac{2}{9}, r=\left(\frac{-1}{3}\times\frac{9}{2}\right)=\frac{-3}{2}$ and $n=6$.

$$\therefore \quad S_6 = \frac{\frac{2}{9}\times\left\{1-\left(\frac{-3}{2}\right)^6\right\}}{\left(1+\frac{3}{2}\right)} = \frac{\frac{2}{9}\times\left\{1-\frac{729}{64}\right\}}{\left(\frac{5}{2}\right)} = \left(\frac{2}{9}\times\frac{2}{5}\right)\times\left(\frac{-665}{64}\right) = \frac{-133}{144}.$$

2. (i) $a = \sqrt{7}$ and $r = \dfrac{\sqrt{21}}{\sqrt{7}} = \sqrt{3}$.

$$\therefore \quad S_n = \frac{\sqrt{7}\times\{(\sqrt{3})^n - 1\}}{(\sqrt{3}-1)} \times \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \frac{\sqrt{7}}{2}\cdot(\sqrt{3}+1)(3^{n/2}-1).$$

(ii) $a = 1$ and $r = \dfrac{-1}{3}$.

$$\therefore \quad S_n = \frac{1\times\left\{1-\left(\frac{-1}{3}\right)^n\right\}}{\left(1+\frac{1}{3}\right)} = \frac{3}{4}\times\left\{1-\frac{(-1)^n}{3^n}\right\} = \frac{\{3^n-(-1)^n\}}{4\times 3^{(n-1)}}.$$

(iii) $A = 1$ and $R = -a$.

$$\therefore \quad S_n = \frac{A\times\{1-R^n\}}{(1-R)} = \frac{1\times\{1-(-a)^n\}}{(1+a)} = \frac{\{1-(-a)^n\}}{(1+a)}.$$

(iv) $a = x^3$ and $r = \dfrac{x^5}{x^3} = x^2$.

$$\therefore \quad S_n = \frac{x^3\times\{(x^2)^n-1\}}{(x^2-1)} = \frac{x^3\times(x^{2n}-1)}{(x^2-1)}.$$

(v) Given expression

$$= \{x^2+x^4+x^6+\ldots \text{ to } n \text{ terms}\} + \{xy+x^2y^2+x^3y^3+\ldots \text{to } n \text{ terms}\}$$

$$= x^2(1+x^2+x^4+\ldots \text{to } n \text{ terms}\} + xy(1+xy+x^2y^2+\ldots \text{to } n \text{ terms}\}$$

$$= \frac{x^2\times 1\times\{(x^2)^n-1\}}{(x^2-1)} + \frac{xy\times 1\times\{(xy)^n-1\}}{(xy-1)} = \frac{x^2(x^{2n}-1)}{(x^2-1)} + \frac{xy(x^ny^n-1)}{(xy-1)}.$$

3. (i) Given sum

$$= \left(x+\frac{1}{x}\right)^2 + \left(x^2+\frac{1}{x^2}\right)^2 + \left(x^3+\frac{1}{x^3}\right)^2 + \ldots + \left(x^n+\frac{1}{x^n}\right)^2$$

$$= \left(x^2+\frac{1}{x^2}+2\right) + \left(x^4+\frac{1}{x^4}+2\right) + \left(x^6+\frac{1}{x^6}+2\right) + \ldots + \left(x^{2n}+\frac{1}{x^{2n}}+2\right)$$

$$= (x^2+x^4+x^6+\ldots+x^{2n}) + \left(\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\ldots+\frac{1}{x^{2n}}\right) + (2+2+\ldots n \text{ times})$$

$$= \frac{x^2[(x^2)^n-1]}{(x^2-1)} + \frac{\frac{1}{x^2}\cdot\left\{1-\left(\frac{1}{x^2}\right)^n\right\}}{\left(1-\frac{1}{x^2}\right)} + 2n = \frac{x^2(x^{2n}-1)}{(x^2-1)} + \frac{(x^{2n}-1)}{x^{2n}(x^2-1)} + 2n$$

$$= x^2\left(\frac{x^{2n}-1}{x^2-1}\right) + \frac{1}{x^{2n}}\left(\frac{x^{2n}-1}{x^2-1}\right) + 2n = \left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right) + 2n.$$

(ii) Multiplying and dividing each term by $(x-y)$, we get the given sum as

$$\frac{1}{(x-y)}\cdot\{(x^2-y^2)+(x^3-y^3)+(x^4-y^4)+\ldots \text{ to } n \text{ terms}\}$$

$$=\frac{1}{(x-y)}\cdot\{(x^2+x^3+x^4+\ldots \text{ to } n \text{ terms})-(y^2+y^3+y^4+\ldots \text{ to } n \text{ terms})\}$$

$$=\frac{1}{(x-y)}\cdot\left\{x^2\left(\frac{x^n-1}{x-1}\right)-y^2\left\{\frac{y^n-1}{y-1}\right\}\right\}.$$

4. Given sum $=\left\{\frac{3}{5}+\frac{3}{5^3}+\frac{3}{5^5}+\ldots \text{ to } n \text{ terms}\right\}+\left\{\frac{4}{5^2}+\frac{4}{5^4}+\ldots \text{ to } n \text{ terms}\right\}$

$$=\frac{\frac{3}{5}\times\left\{1-\left(\frac{1}{5^2}\right)^n\right\}}{\left(1-\frac{1}{5^2}\right)}+\frac{\frac{4}{5^2}\times\left\{1-\left(\frac{1}{5^2}\right)^n\right\}}{\left(1-\frac{1}{5^2}\right)}$$

$$=\left(\frac{3}{5}\times\frac{25}{24}\right)\frac{(5^{2n}-1)}{5^{2n}}+\left(\frac{4}{25}\times\frac{25}{24}\right)\frac{(5^{2n}-1)}{5^{2n}}$$

$$=\left(\frac{5}{8}+\frac{1}{6}\right)\left\{\frac{5^{2n}-1}{5^{2n}}\right\}=\frac{19}{24}\cdot\left\{\frac{5^{2n}-1}{5^{2n}}\right\}.$$

5. (i) $\sum_{n=1}^{10}(2+3^n)=(2+3^1)+(2+3^2)+(2+3^3)+\ldots+(2+3^{10})$

$$=(2+2+2+\ldots \text{ taken 10 times})+(3+3^2+3^3+\ldots+3^{10})$$

$$=(2\times 10)+\left\{3\times\left(\frac{3^{10}-1}{3-1}\right)\right\}$$

$$=20+\left(\frac{59049-1}{2}\right)=(20+29524)=29544.$$

(ii) $\sum_{k=1}^{n}[2^k+3^{(k-1)}]=(2^1+3^0)+(2^2+3^1)+(2^3+3^2)+\ldots \text{ up to } n \text{ terms}$

$$=(2^1+2^2+2^3+\ldots \text{ to } n \text{ terms})+(1+3+3^2+\ldots \text{ to } n \text{ terms})$$

$$=2\left(\frac{2^n-1}{2-1}\right)+1\cdot\left(\frac{3^n-1}{3-1}\right)$$

$$=2(2^n-1)+\frac{1}{2}(3^n-1)=2^{(n+1)}-2+\frac{1}{2}\cdot 3^n-\frac{1}{2}$$

$$=2^{(n+1)}+\frac{1}{2}\cdot 3^n-\frac{5}{2}=2^{(n+1)}+\frac{1}{2}(3^n-5)=\frac{1}{2}[2^{(n+2)}+3^n-5].$$

(iii) $\sum_{n=1}^{8}5^n=[5+5^2+5^3+\ldots \text{ to } 8 \text{ terms}]$

$$=5\left(\frac{5^8-1}{5-1}\right)=\frac{5(390625-1)}{4}=\frac{5\times 390624}{4}$$

$$=(5\times 97656)=488280.$$

6. (i) Given sum $=8\times[1+11+111+\ldots \text{ to } n \text{ terms}]$

$$=\frac{8}{9}\times[9+99+999+\ldots \text{ to } n \text{ terms}]$$

$$=\frac{8}{9}\times[(10-1)+(10^2-1)+(10^3-1)+\ldots \text{ to } n \text{ terms}]$$

$$= \frac{8}{9} \times \{(10 + 10^2 + 10^3 + \ldots \text{ to } n \text{ terms}) - n\}$$

$$= \frac{8}{9} \times \left\{10 \times \left(\frac{10^n - 1}{10 - 1}\right) - n\right\} = \frac{8}{81}[10^{(n-1)} - 10 - 9n].$$

(iii) Given series $= \frac{7}{10} + \frac{77}{100} + \frac{777}{1000} + \ldots$ to n terms

$$= 7 \times \left\{\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + \ldots \text{ to } n \text{ terms}\right\}$$

$$= \frac{7}{9} \times \left\{\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots \text{ to } n \text{ terms}\right\}$$

$$= \frac{7}{9} \times \left\{\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \ldots \text{ to } n \text{ terms}\right\}$$

$$= \frac{7}{9} \times \left\{n - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots \text{ to } n \text{ terms}\right)\right\}$$

$$= \frac{7}{9} \times \left\{n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{\left(1 - \frac{1}{10}\right)}\right\} = \frac{7}{81} \times \left(9n - 1 + \frac{1}{10^n}\right).$$

7. $S_n = (2^n - 1)$ and $S_{n-1} = [2^{(n-1)} - 1]$.

$\therefore \quad T_n = (S_n - S_{n-1}) = [2^n - 2^{(n-1)}] \Rightarrow T_1 = 1, T_2 = 2, T_3 = 4, T_4 = 8, \ldots$.

Clearly, 1, 2, 4, 8, … is a GP with common ratio = 2.

15. Let us consider a GP with first term a and common ratio r and having $2n$ terms. Then,

$$T_1 + T_2 + T_3 + T_4 + \ldots + T_{2n} = 5(T_1 + T_3 + T_5 + \ldots + T_{2n-1})$$

$$\Rightarrow \quad a + ar + ar^2 + \ldots + ar^{(2n-1)} = 5[a + ar^2 + ar^4 + \ldots + ar^{(2n-2)}]$$

$$\Rightarrow \quad a\left(\frac{1 - r^{2n}}{1 - r}\right) = 5a\left[\frac{1 - (r^2)^n}{1 - r^2}\right] \Rightarrow 1 + r = 5 \Rightarrow r = 4.$$

16. Required ratio $= S_n : (S_{2n} - S_n)$.

Now, $S_n = \frac{a(1 - r^n)}{(1 - r)}$ and $S_{2n} = \frac{a(1 - r^{2n})}{(1 - r)} = \frac{a(1 - r^n)(1 + r^n)}{(1 - r)}$

$$\therefore \quad (S_{2n} - S_n) = \frac{a(1 - r^n)(1 + r^n)}{(1 - r)} - \frac{a(1 - r^n)}{(1 - r)} = \frac{a(1 - r^n)(1 + r^n - 1)}{(1 - r)} = \frac{a(1 - r^n)r^n}{(1 - r)}.$$

Hence, the required ratio is $\frac{1}{r^n}$.

WORD PROBLEMS ON GP

EXAMPLE 1 *A man has 2 parents, 4 grandparents, 8 great-grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own.*

SOLUTION Required number of ancestors

$$= 2 + 4 + 6 + 8 + \ldots \text{ up to 10 terms}$$

$$= 2 \times \frac{(2^{10} - 1)}{(2 - 1)} = 2(2^{10} - 1) \qquad [\text{GP with } a = 2, r = 2 \text{ and } n = 10]$$

$= 2(1024 - 1) = (2 \times 1023) = 2046.$

Hence, the required number of the man's ancestors is 2046.

EXAMPLE 2 *A man writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with the instruction that they move the chain similarly. Assuming that the chain is not broken and it costs* ` 4 *to mail one letter, find the amount spent on postage when* 6*th set of letters is mailed.*

SOLUTION Successive number of letters are 4, 16, 64, … .

This is a GP with $a = 4$ and $r = 4$.

Number of letters in the 6th set $= ar^{(6-1)} = (4 \times 4^5) = 4^6 = 4096.$

Cost of postage on these leters $=$ ` $(4096 \times 4) =$ ` 16384.

EXAMPLE 3 *What will* ` 10000 *amount to in* 4 *years after its deposit in a bank which pays annual interest at the rate of* 10% *per annum compounded annually?*

SOLUTION Original sum = ` 10000.

Amount after 1 year = ` $10000 +$ ` $\left(10000 \times \frac{10}{100} \times 1\right) =$ ` 11000,

amount after 2 years = ` $11000 +$ ` $\left(11000 \times \frac{10}{100} \times 1\right) =$ ` 12100,

amount after 3 years = ` $12100 +$ ` $\left(12100 \times \frac{10}{100} \times 1\right) =$ ` 13310,

and so on.

Thus, these amounts form a GP

$10000, 11000, 12100, 13310, \ldots$

such that $\frac{11000}{10000} = \frac{12100}{11000} = \frac{13310}{12100} = \frac{11}{10}$.

In this GP, we have $a = 10000$, $r = \frac{11}{10}$.

Amount after 4 years $= T_5 = ar^{(5-1)} = ar^4$

$=$ ` $\left[10000 \times \left(\frac{11}{10}\right)^4\right] =$ ` 14641.

Hence, the amount after 4 years is ` 14641.

Alternative method

Here, $P =$ ` 10000, $R = 10\%$ p.a. and $T = 4$ years.

$\therefore$ amount after 4 years = ` $\left\{P \times \left(1 + \frac{R}{100}\right)^T\right\}$

= ` $\left\{10000 \times \left(1 + \frac{10}{100}\right)^4\right\}$

= ` $\left\{10000 \times \left(\frac{11}{10}\right)^4\right\} =$ ` 14641.

Hence, the amount after 4 years is ` 14641.

EXAMPLE 4 *A manufacturer reckons that the value of a machine which costs him ` 125000 will depreciate each year by 20%. Find the estimated value of the machine at the end of 5 years.*

SOLUTION **Short-cut method**

Value of the machine after 5 years $= ` \left\{125000 \times \left(1 - \frac{20}{100}\right)^5\right\}$

$= ` \left\{125000 \times \left(\frac{4}{5}\right)^5\right\}$

$= ` \left(\frac{125000 \times 1024}{3125}\right) = ` \ 40960.$

Hence, the value of the machine after 5 years is ` 40960.

Alternative method

Initial value of the machine = ` 125000.

Value of the machine after 1 year = 80% of ` 125000

$= ` \left(125000 \times \frac{80}{100}\right) = ` \ 100000.$

Value of the machine after 2 years = 80% of ` 100000

$= ` \left(100000 \times \frac{80}{100}\right) = ` \ 80000.$

Thus, the yearwise values of the machine are

` 125000, ` 100000, ` 80000, … .

Here, $\frac{100000}{125000} = \frac{4}{5}, \frac{80000}{100000} = \frac{4}{5}$, etc.

Thus, these amounts form a GP with $a = 125000$ and $r = \frac{4}{5}$.

Value after 5 years = 6th term $= T_6 = ar^{(6-1)} = ar^5$

$= ` \left\{125000 \times \left(\frac{4}{5}\right)^5\right\}$

$= ` \left(\frac{125000 \times 1024}{3125}\right) = ` \ 40960.$

EXAMPLE 5 *The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria would be present at the end of 2nd hour, 4th hour and nth hour?*

SOLUTION The bacteria present in the culture originally, at the end of 1st hour, at the end of 2nd hour, at the end of 3rd hour and so on, are

30, 60, 120, 240, … .

This is a GP with $a = 30$ and $r = \frac{60}{30} = 2$.

Number of bacteria at the end of 2nd hour

$= T_3 = ar^2 = (30 \times 2^2) = 120.$

Number of bacteria at the end of 4th hour

$= T_5 = ar^4 = (30 \times 2^4) = 480.$

Number of bacteria at the end of nth hour

$$= T_{n+1} = ar^{(n+1-1)} = 30 \times 2^n.$$

EXAMPLE 6 *The inventor of the chessboard suggested a reward of one grain of wheat for the first square; 2 grains for the second; 4 grains for the third; and so on, doubling the number of grains for subsequent squares. How many grains would have to be given to the inventor? (Note that there are 64 squares in the chessboard.)*

SOLUTION Required number of grains

$$= 1 + 2^1 + 2^2 + 2^3 + \ldots \text{ to } 64 \text{ terms}$$
$$= 1 + (2^1 + 2^2 + 2^3 + \ldots 2^{63})$$
$$= 1 + \frac{2(2^{63} - 1)}{(2-1)} = (1 + 2^{64} - 2) = (2^{64} - 1).$$

EXAMPLE 7 *The lengths of three unequal edges of a rectangular solid block are in GP. The volume of the block is* 216 cm^3 *and the total surface area is* 252 cm^2. *Find the length of its longest edge.*

SOLUTION Let the lengths of its edges be $\frac{a}{r}$ cm, a cm and ar cm.

Then, its volume $= \left(\frac{a}{r} \times a \times ar\right) \text{cm}^3 = a^3 \text{ cm}^3$.

$\therefore \quad a^3 = 216 = 6^3 \Rightarrow a = 6.$

So, the edges are $6r$ cm, 6 cm and $\frac{6}{r}$ cm.

$\therefore$ surface area $= 2[lb + bh + lh]$

$$= 2\left[6r \times 6 + 6 \times \frac{6}{r} + 6r \times \frac{6}{r}\right] \text{cm}^2$$
$$= 2\left[36r + \frac{36}{r} + 36\right] \text{cm}^2 = 72 \times \left[r + \frac{1}{r} + 1\right] \text{cm}^2.$$

$\therefore \quad 72 \times \left(r + \frac{1}{r} + 1\right) = 252$

$\Rightarrow \quad 2(r^2 + 1 + r) = 7r \Rightarrow 2r^2 - 5r + 2 = 0$

$\Rightarrow \quad (r-2)(2r-1) = 0 \Rightarrow r = 2 \text{ or } r = \frac{1}{2}.$

Each value of r gives the longest edge = 12 cm.

EXERCISE 12D

1. What will ` 15625 amount to in 3 years after its deposit in a bank which pays annual interest at the rate of 8% per annum, compounded annually?
2. The value of a machine costing ` 80000 depreciates at the rate of 15% per annum. What will be the worth of this machine after 3 years?
3. Three years before the population of a village was 10000. If at the end of each year, 20% of the people migrated to a nearby town, what is its present population?

4. What will ` 5000 amount to in 10 years, compounded annually at 10% per annum? [Given $(1.1)^{10} = 2.594$]
5. A manufacturer reckons that the value of a machine which costs him ` 156250, will depreciate each year by 20%. Find the estimated value at the end of 5 years.
6. The number of bacteria in a certain culture doubles every hour. If there were 50 bacteria present in the culture originally, how many bacteria would be present at the end of (i) 2nd hour, (ii) 5th hour and (iii) nth hour?

Hint At the end of 2nd hour means 3rd term $= ar^2 = (50 \times 2^2)$.

ANSWERS (EXERCISE 12D)

1. ` 19683 **2.** ` 49130 **3.** 5120 **4.** ` 12970

5. ` 51200 **6.** (i) 200 (ii) 1600 (iii) 50×2^n

PROPERTIES OF GP

Let $a, ar, ar^2, ar^3, \ldots$ be a GP. Then,

(i) $\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \frac{1}{ar^3}, \ldots$ is a GP.

(ii) $ka, k(ar), k(ar^2), k(ar^3), \ldots$ is a GP, where $k \neq 0$.

(iii) $\frac{k}{a}, \frac{k}{ar}, \frac{k}{ar^2}, \frac{k}{ar^3}, \ldots$ is a GP, where $k \neq 0$.

(iv) $a^k, (ar)^k, (ar^2)^k, (ar^3)^k, \ldots$ is a GP, where $k \neq 0$.

SOLVED EXAMPLES

EXAMPLE 1 *If a, b, c are in GP, prove that* $\log a, \log b, \log c$ *are in AP.*

SOLUTION Let *a, b, c* be in GP. Then,

$$b^2 = ac \Rightarrow \log b^2 = \log (ac)$$
$$\Rightarrow 2\log b = \log a + \log c$$
$$\Rightarrow \log a, \log b, \log c \text{ are in AP.}$$

EXAMPLE 2 *If a, b, c, d are in GP then prove that* $a+b, b+c, c+d$ *are also in GP.*

SOLUTION Let *a, b, c, d* be in GP with common ratio *r*. Then,

$$b = ar, c = ar^2 \text{ and } d = ar^3.$$
$$\therefore \quad (a+b) = (a+ar) = a(1+r),$$
$$(b+c) = (ar+ar^2) = ar(1+r),$$
$$(c+d) = (ar^2+ar^3) = ar^2(1+r).$$
$$\therefore \quad (b+c)^2 = a^2r^2(1+r)^2 \text{ and } (a+b)(c+d) = a^2r^2(1+r)^2.$$

Consequently, $(b+c)^2 = (a+b)(c+d)$.

Hence, $(a+b), (b+c)$ and $(c+d)$ are in GP.

EXAMPLE 3 *If a, b, c, d are in GP then prove that* $a^n + b^n, b^n + c^n, c^n + d^n$ *are also in GP.*

SOLUTION Let *a, b, c, d* be in GP with common ratio = r. Then,

$$b = ar,\ c = ar^2 \text{ and } d = ar^3.$$

$$\therefore \quad (a^n + b^n) = (a^n + a^n r^n) = a^n(1 + r^n),$$

$$(b^n + c^n) = (a^n r^n + a^n r^{2n}) = a^n r^n (1 + r^n)$$

and $\quad (c^n + d^n) = (a^n r^{2n} + a^n r^{3n}) = a^n r^{2n}(1 + r^n).$

$$\therefore \quad (b^n + c^n)^2 = a^{2n} r^{2n}(1 + r^n)^2 \text{ and } (a^n + b^n)(c^n + d^n) = a^{2n} r^{2n}(1 + r^n)^2.$$

Consequently, $(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$.

Hence, $(a^n + b^n), (b^n + c^n)$ and $(c^n + d^n)$ are in GP.

EXAMPLE 4 *If* $(a^2 + b^2), (ab + bc), (b^2 + c^2)$ *are in GP then prove that a, b, c are also in GP.*

SOLUTION Let $(a^2 + b^2), (ab + bc)$ and $(b^2 + c^2)$ be in GP. Then,

$$(ab + bc)^2 = (a^2 + b^2)(b^2 + c^2)$$

$$\Rightarrow \quad a^2b^2 + b^2c^2 + 2ab^2c = a^2b^2 + a^2c^2 + b^4 + b^2c^2$$

$$\Rightarrow \quad b^4 + a^2c^2 - 2ab^2c = 0$$

$$\Rightarrow \quad (b^2 - ac)^2 = 0 \ \Rightarrow \ b^2 - ac = 0 \ \Rightarrow \ b^2 = ac.$$

Hence, *a, b, c* are in GP.

EXAMPLE 5 *If mth, nth and pth terms of a GP form three consecutive terms of a GP then prove that m, n, p are three consecutive terms of a AP.*

SOLUTION Let *a* be the first term and *r* be the common ratio of the given GP. Then,

$$T_m = ar^{(m-1)},\ T_n = ar^{(n-1)} \text{ and } T_p = ar^{(p-1)}.$$

Since T_m, T_n, T_p are in GP, we have

$$T_n^2 = T_m \times T_p$$

$$\Rightarrow \quad [ar^{(n-1)}]^2 = [ar^{(m-1)} \times ar^{(p-1)}]$$

$$\Rightarrow \quad a^2 r^{(2n-2)} = a^2 r^{(m+p-2)}$$

$$\Rightarrow \quad r^{(2n-2)} = r^{(m+p-2)}$$

$$\Rightarrow \quad 2n - 2 = m + p - 2$$

$$\Rightarrow \quad 2n = m + p \ \Rightarrow \ m, n, p \text{ are in AP.}$$

Hence, *m, n, p* are three consecutive terms of an AP.

EXAMPLE 6 *If the* 4*th,* 10*th and* 16*th terms of a GP are x, y, z respectively, prove that x, y, z are in GP.*

SOLUTION Let *a* be the first term and *r* be the common ratio of the given GP.

Then, $T_4 = x,\ T_{10} = y$ and $T_{16} = z$.

$\Rightarrow \quad ar^3 = x,\ ar^9 = y$ and $ar^{15} = z.$

$\therefore \quad y^2 = (ar^9)^2 = a^2r^{18}$ *and* $xz = (ar^3)(ar^{15}) = a^2r^{18}.$

Consequently, we have $y^2 = xz.$

Hence, *x*, *y*, *z* are in GP.

EXAMPLE 7 *Three numbers are in AP and their sum is* 15. *If* 1, 3, 9 *be added to them respectively, they form a GP. Find the numbers.*

SOLUTION Let the required numbers be $(a-d)$, a and $(a+d)$. Then,

$$(a-d)+a+(a+d) = 15 \ \Rightarrow\ 3a = 15 \ \Rightarrow\ a = 5.$$

So, the numbers are $(5-d)$, 5 and $(5+d)$.

Adding 1, 3, 9 respectively to these numbers, we get the numbers

$(6-d)$, 8 and $(14+d)$.

These numbers are in GP.

$$\therefore \quad 8^2 = (6-d)(14+d) \ \Rightarrow\ 84 - 8d - d^2 = 64$$
$$\Rightarrow\ d^2 + 8d - 20 = 0$$
$$\Rightarrow d^2 + 10d - 2d - 20 = 0$$
$$\Rightarrow\ d(d+10) - 2(d+10) = 0$$
$$\Rightarrow\ (d+10)(d-2) = 0$$
$$\Rightarrow\ d = -10 \ \text{ or } \ d = 2.$$

So, the required numbers are $(3, 5, 7)$ or $(15, 5, -5)$.

EXERCISE 12E

1. If *p*, *q*, *r* are in AP then prove that *p*th, *q*th and *r*th terms of any GP are in GP.
2. If *a*, *b*, *c* are in GP then show that $\log a^n, \log b^n, \log c^n$ are in AP.
3. If *a*, *b*, *c* are in GP then show that $\frac{1}{\log_a m}, \frac{1}{\log_b m}, \frac{1}{\log_c m}$ are in AP.
4. Find the values of *k* for which $k+12$, $k-6$ and 3 are in GP.
5. Three numbers are in AP and their sum is 15. If 1, 4, 19 be added to them respectively then they are in GP. Find the numbers.
6. Three numbers are in AP and their sum is 21. If the second number is reduced by 1 and the third is increased by 1, we obtain three numbers in GP. Find the numbers.
7. The sum of three numbers in GP is 56. If 1, 7, 21 be subtracted from them respectively, we obtain the numbers in AP. Find the numbers.
8. If *a*, *b*, *c* are in GP, prove that $\frac{a^2+ab+b^2}{ab+bc+ca} = \frac{b+a}{c+b}$.
9. If $(a-b)$, $(b-c)$, $(c-a)$ are in GP then prove that $(a+b+c)^2 = 3(ab+bc+ca)$.
10. If *a*, *b*, *c* are in GP, prove that
    (i) $a(b^2+c^2) = c(a^2+b^2)$
    (ii) $\frac{1}{(a^2-b^2)} + \frac{1}{b^2} = \frac{1}{(b^2-c^2)}$

(iii) $(a+2b+2c)(a-2b+2c)=a^2+4c^2$

(iv) $a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=a^3+b^3+c^3.$

11. If *a, b, c, d* are in GP, prove that

(i) $(b+c)(b+d)=(c+a)(c+d)$

(ii) $\frac{ab-cd}{b^2-c^2}=\frac{a+c}{b}$

(iii) $(a+b+c+d)^2=(a+b)^2+2(b+c)^2+(c+d)^2.$

12. If *a, b, c* are in GP, prove that $\frac{1}{(a+b)}, \frac{1}{2b}, \frac{1}{b+c}$ are in AP.

13. If *a, b, c* are in GP, prove that a^2, b^2, c^2 are in GP.

14. If *a, b, c* are in GP, prove that a^3, b^3, c^3 are in GP.

15. If *a, b, c* are in GP, prove that $(a^2+b^2), (ab+bc), (b^2+c^2)$ are in GP.

16. If *a, b, c, d* are in GP, prove that $(a^2-b^2), (b^2-c^2), (c^2-d^2)$ are in GP.

17. If *a, b, c, d* are in GP then prove that

$\frac{1}{(a^2+b^2)}, \frac{1}{(b^2+c^2)}, \frac{1}{(c^2+d^2)}$ are in GP.

18. If $(p^2+q^2), (pq+qr), (q^2+r^2)$ are in GP then prove that *p, q, r* are in GP.

19. If *a, b, c* are in AP and *a, b, d* are in GP, show that $a, (a-b)$ and $(d-c)$ are in GP.

20. If *a, b, c* are in AP, and *a, x, b* and *b, y, c* are in GP then show that x^2, b^2, y^2 are in AP.

ANSWERS (EXERCISE 12E)

4. $(k=0)$ or $(k=15)$ **5.** (2, 5, 8) or (26, 5, –16) **6.** (12, 7, 2) or (3, 7, 11)

7. 8, 16, 32

HINTS TO SOME SELECTED QUESTIONS

1. We have, $2q=p+r.$

Let T_p, T_q, T_r be the given terms of a GP with first term $=A$ and common ratio $=R$. Then,

$$(T_q)^2=\{AR^{(q-1)}\}^2=A^2R^{(2q-2)}$$

and $(T_p\times T_r)=\{AR^{(p-1)}\times AR^{(r-1)}\}=A^2R^{(p+r-2)}=A^2R^{(2q-2)}.$

$\therefore$ $(T_q)^2=(T_p\times T_r)$ and hence T_p, T_q, T_r are in GP.

2. *a, b, c* are in GP $\Rightarrow$ $b^2=ac \Rightarrow (b^2)^n=(ac)^n \Rightarrow b^{2n}=a^nc^n.$

$\therefore$ $2\log b^n=\log a^n+\log c^n \Rightarrow \log a^n, \log b^n, \log c^n$ are in AP.

3. *a, b, c* are in GP $\Rightarrow$ $b^2=ac$

$\therefore$ $2\log_m b=\log_m a+\log_m c \Rightarrow 2\times\frac{\log b}{\log m}=\frac{\log a}{\log m}+\frac{\log c}{\log m}$

$\therefore \quad 2\times\frac{1}{\log_b m}=\frac{1}{\log_a m}+\frac{1}{\log_c m} \Rightarrow \frac{1}{\log_a m},\frac{1}{\log_b m},\frac{1}{\log_c m}$ are in AP.

4. $\frac{k-6}{k+12}=\frac{3}{k-6} \Rightarrow (k-6)^2=3k+36$

$\Rightarrow k^2-12k=3k \Rightarrow k^2-15k=0 \Rightarrow k(k-15)=0$

$\therefore \quad k=0$ or $k=15$.

6. Let the required numbers be $(a-d), a, (a+d)$. Then, $3a=21 \Rightarrow a=7$.

So, these numbers are $(7-d), 7, (7+d)$.

$\therefore \quad (7-d), 6$ and $(8+d)$ are in GP.

So, $\frac{6}{(7-d)}=\frac{(8+d)}{6} \Rightarrow (7-d)(8+d)=36 \Rightarrow d^2+d-20=0$

$\Rightarrow (d+5)(d-4)=0 \Rightarrow d=-5$ or $d=4$.

Hence, the required numbers are (12, 7, 2) or (3, 7, 11).

8. Let r be the common ratio of the given GP. Then,

$b=ar$ and $c=ar^2$.

$\therefore \quad \text{LHS}=\frac{a^2+a^2r+a^2r^2}{a^2r+a^2r^3+a^2r^2}=\frac{a^2(1+r+r^2)}{a^2r(1+r+r^2)}=\frac{1}{r}$

$\text{RHS}=\frac{ar+a}{ar^2+ar}=\frac{a(1+r)}{ar(1+r)}=\frac{1}{r}$.

Hence, LHS = RHS.

9. $\frac{(b-c)}{(a-b)}-\frac{(c-a)}{(b-c)} \Rightarrow (b-c)^2=(a-b)(c-a)$

$\Rightarrow b^2+c^2-2bc=ac-a^2-bc+ab$

$\Rightarrow a^2+b^2+c^2=ab+bc+ac$

$\Rightarrow (a+b+c)^2=3(ab+bc+ac)$ [adding $2(ab+bc+ac)$ on both sides].

12. Let $b=ar$ and $c=ar^2$. Then,

$\frac{1}{a+b}=\frac{1}{a+ar}=\frac{1}{a(1+r)}, \frac{1}{2b}=\frac{1}{2ar}$, and $\frac{1}{(b+c)}=\frac{1}{ar(1+r)}$.

$\therefore \quad \frac{1}{(a+b)}+\frac{1}{(b+c)}=\frac{1}{a(1+r)}+\frac{1}{ar(1+r)}=\frac{(1+r)}{ar(1+r)}=\frac{1}{ar}=2\times\left(\frac{1}{2b}\right)$.

Hence, $\frac{1}{(a+b)}, \frac{1}{2b}, \frac{1}{(b+c)}$ are in AP.

13. Let $b=ar$ and $c=ar^2$. Then

$a^2c^2=a^2(ar^2)^2=a^2\times a^2r^4=a^4r^4=(b^2)^2$.

19. $(a+c)=2b$ and $b^2=ad$.

We have to prove that $(a-b)^2=a(d-c)$.

Now, $(a-b)^2=a^2+b^2-2ab=a^2+ad-a(a+c)$

$\Rightarrow \quad (a-b)^2=ad-ac=a(d-c)$.

Hence, $a, (a-b), (d-c)$ are in GP.

20. $x^2=ab, y^2=bc$ and $a+c=2b$.

$\therefore \quad x^2+y^2=(ab+bc)=b(a+c)=2b^2$.

Hence, x^2, b^2, y^2 are in AP.

GEOMETRIC MEAN

GEOMETRIC MEAN *Let a and b be two given numbers. We say that G is the geometric mean (GM) between a and b if a, G, b are in GP.*

G is the GM between a and $b \Leftrightarrow a, G, b$ are in GP

$$\Leftrightarrow \frac{G}{a} = \frac{b}{G}$$

$$\Leftrightarrow G^2 = ab \Leftrightarrow G = \sqrt{ab}.$$

So, *GM between a and* $b = \sqrt{ab}$.

REMARKS (i) The GM between two positive numbers is positive.

(ii) The GM between two negative numbers is negative.

(iii) The GM between two numbers of opposite signs does not exist.

SOLVED EXAMPLES

EXAMPLE 1 *Find the geometric mean between*

(i) 6 *and* 24 *(ii)* –9 *and* –25 *(iii)* –6 *and* 9

SOLUTION (i) The GM between 6 and $24 = \sqrt{6 \times 24} = \sqrt{144} = 12.$

(ii) The GM between –9 and $-25 = \sqrt{(-9) \times (-25)} = \sqrt{225} = -15.$

(iii) Since the GM between two numbers of opposite signs does not exist, so the GM between –6 and 9 does not exist.

EXAMPLE 2 *Find two positive numbers a and b whose AM and GM are* 34 *and* 16 *respectively.*

SOLUTION We have

$$\frac{a+b}{2} = 34 \text{ and } \sqrt{ab} = 16$$

$$\Rightarrow \quad a + b = 68 \text{ and } ab = 256$$

$$\Rightarrow \quad (a-b) = \sqrt{(a+b)^2 - 4ab} = \sqrt{(68)^2 - 4 \times 256} = \sqrt{3600} = \pm 60$$

$$\Rightarrow \quad a + b = 68 \text{ and } a - b = \pm 60$$

$$\Rightarrow \quad (a + b = 68, a - b = 60) \text{ or } (a + b = 68, a - b = -60)$$

$$\Rightarrow \quad (a = 64, b = 4) \text{ or } (a = 4, b = 64)$$

Hence, the required numbers are $(a = 64, b = 4)$ or $(a = 4, b = 64)$.

EXAMPLE 3 *Find the value of n so that* $\frac{a^{(n+1)} + b^{(n+1)}}{a^n + b^n}$ *may be the geometric mean between a and b, where* $a \neq b$.

SOLUTION $\frac{a^{(n+1)} + b^{(n+1)}}{a^n + b^n}$ is GM between a and b

$$\Rightarrow \quad \frac{a^{(n+1)} + b^{(n+1)}}{a^n + b^n} = a^{\frac{1}{2}} b^{\frac{1}{2}}$$

$$\Rightarrow \quad [a^{(n+1)} + b^{(n+1)}] = (a^n + b^n)\left(a^{\frac{1}{2}} b^{\frac{1}{2}}\right)$$

$$\Rightarrow \quad [a^{(n+1)} + b^{(n+1)}] = a^{\left(n+\frac{1}{2}\right)} b^{\frac{1}{2}} + a^{\frac{1}{2}} b^{\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow \quad \left\{a^{(n+1)} - a^{\left(n+\frac{1}{2}\right)} b^{\frac{1}{2}}\right\} = a^{\frac{1}{2}} b^{\left(n+\frac{1}{2}\right)} - b^{(n+1)}$$

$$\Rightarrow \quad a^{\left(n+\frac{1}{2}\right)}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = b^{\left(n+\frac{1}{2}\right)}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)$$

$$\Rightarrow \quad a^{\left(n+\frac{1}{2}\right)} = b^{\left(n+\frac{1}{2}\right)} \qquad \left[\because \left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) \neq 0, \text{ since } a \neq b\right]$$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^{\left(n+\frac{1}{2}\right)} = 1 = \left(\frac{a}{b}\right)^{0}$$

$$\Rightarrow \quad n + \frac{1}{2} = 0 \Rightarrow n = -\frac{1}{2}.$$

Hence, $n = -\frac{1}{2}$.

EXAMPLE 4 *If A and G are respectively the arithmetic and geometric means between two distinct positive numbers a and b then prove that* $A > G$.

SOLUTION We have

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

$$\Rightarrow \quad A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0$$

$$\Rightarrow \quad A > G.$$

Hence, AM > GM.

EXAMPLE 5 *If A and G are respectively the arithmetic and geometric means between two positive numbers a and b then prove that the quadratic equation having a, b as its roots is given by* $x^2 - 2Ax + G^2 = 0$.

SOLUTION We have

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}.$$

The equation having *a, b* as its roots is given by

$$x^2 - (a+b)x + ab = 0$$

$$\Rightarrow \quad x^2 - 2Ax + G^2 = 0 \qquad \left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right].$$

Hence, the quadratic equation having *a, b* as its roots is $x^2 - 2Ax + G^2 = 0$.

EXAMPLE 6 *If A and G be the AM and GM between two positive numbers then prove that the numbers are* $A \pm \sqrt{A^2 - G^2}$.

SOLUTION Let the required numbers be *a* and *b*. Then,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}.$$

The quadratic equation having *a, b* as its roots is given by

$$x^2 - (a+b)x + ab = 0$$

$$\Rightarrow \quad x^2 - 2Ax + G^2 = 0$$

$$\Rightarrow \quad x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2} \Rightarrow x = A \pm \sqrt{A^2 - G^2}.$$

Hence, the required numbers are $A \pm \sqrt{A^2 - G^2}$.

EXAMPLE 7 *If x, y, z are distinct positive numbers then prove that* $(x+y)(y+z)(z+x) > 8xyz$.

SOLUTION Using AM > GM, we have

$$\frac{x+y}{2} > \sqrt{xy}, \quad \frac{y+z}{2} > \sqrt{yz} \text{ and } \frac{z+x}{2} > \sqrt{zx}$$

$$\Rightarrow \quad x+y > 2\sqrt{xy}, y+z > 2\sqrt{yz} \text{ and } z+x > 2\sqrt{zx}$$

$$\Rightarrow \quad (x+y)(y+z)(z+x) > 2\sqrt{xy} \times 2\sqrt{yz} \times 2\sqrt{zx}$$

$$\Rightarrow \quad (x+y)(y+z)(z+x) > 8xyz.$$

Hence, $(x+y)(y+z)(z+x) > 8xyz$.

EXAMPLE 8 *If a, b, c, d are four distinct positive numbers in GP then prove that* $a+d > b+c$.

SOLUTION *a, b, c, d* are in GP

$\Rightarrow$ *a, b, c* are in GP

$\Rightarrow$ *b* is the GM between *a* and *c*.

Also, we know that $\frac{a+c}{2}$ is the AM between *a* and *c*.

But, AM > GM.

$$\therefore \quad \frac{a+c}{2} > b \Rightarrow a+c > 2b. \qquad \ldots \text{(i)}$$

Again, *a, b, c, d* are in GP

$\Rightarrow$ *b, c, d* are in GP

$\Rightarrow$ *c* is the GM between *b* and *d*.

Also, we know that $\frac{b+d}{2}$ is the AM between *b* and *d*.

But, AM > GM.

$$\therefore \quad \frac{b+d}{2} > c \Rightarrow b+d > 2c. \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$a+c+b+d > 2b+2c.$$

Hence, $a+d > b+c$.

EXAMPLE 9 *The sum of two numbers is* 6 *times their geometric mean. Show that the numbers are in the ratio* $(3+2\sqrt{2}) : (3-2\sqrt{2})$.

SOLUTION Let the required numbers be *a* and *b*. Then,

$$a+b = 6\sqrt{ab}$$

$$\Rightarrow \quad \frac{a+b}{2\sqrt{ab}} = \frac{3}{1}$$

$$\Rightarrow \quad \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3+1}{3-1} = \frac{2}{1} \qquad \text{[by componendo and dividendo]}$$

$$\Rightarrow \quad \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2}=\frac{(\sqrt{2})^2}{1^2} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{2}}{1}$$

$$\Rightarrow \quad \frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})}=\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

[by componendo and dividendo]

$$\Rightarrow \quad \frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{2}+1}{\sqrt{2}-1} \Rightarrow \left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2=\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)^2}$$

$$\Rightarrow \quad \frac{a}{b}=\frac{3+2\sqrt{2}}{3-2\sqrt{2}}.$$

Hence, $a:b=(3+2\sqrt{2}):(3-2\sqrt{2})$.

EXAMPLE 10 *If the AM and GM of the roots of a quadratic equation be* 8 *and* 5 *respectively then obtain the equation.*

SOLUTION Let the required roots be *a* and *b*.

Then, $\text{AM}=\frac{a+b}{2} \Rightarrow \frac{a+b}{2}=8 \Rightarrow a+b=16$ … (i)

and $\text{GM}=\sqrt{ab} \Rightarrow \sqrt{ab}=5 \Rightarrow ab=25.$ … (ii)

$\therefore$ sum of roots = 16 and product of roots = 25.

Hence, the required equation is $x^2-16x+25=0$.

EXAMPLE 11 *If the AM and GM of two positive numbers a and b are in the ratio* $m:n$, *show that*

$$a:b=(m+\sqrt{m^2-n^2}):(m-\sqrt{m^2-n^2}).$$

SOLUTION Let *A* and *G* be respectively the AM and GM of *a* and *b*. Then,

$$A=\frac{a+b}{2} \text{ and } G=\sqrt{ab}$$

$$\Rightarrow \quad a+b=2A \text{ and } G^2=ab.$$

Now, the equation having roots *a* and *b* is

$$x^2-(a+b)x+ab=0$$

$$\Rightarrow \quad x^2-2Ax+G^2=0 \qquad [\because\ a+b=2A \text{ and } ab=G^2]$$

$$\Rightarrow \quad x=\frac{2A\pm\sqrt{4A^2-4G^2}}{2}=A\pm\sqrt{A^2-G^2}$$

$$\Rightarrow \quad a=A+\sqrt{A^2-G^2} \text{ and } b=A-\sqrt{A^2-G^2}.$$

Now, $A:G=m:n$ (given).

Let $A=km$ and $G=kn$ for some constant *k*. Then,

$$\frac{a}{b}=\frac{A+\sqrt{A^2-G^2}}{A-\sqrt{A^2-G^2}}=\frac{km+\sqrt{k^2m^2-k^2n^2}}{km-\sqrt{k^2m^2-k^2n^2}}=\frac{m+\sqrt{m^2-n^2}}{m-\sqrt{m^2-n^2}}.$$

Hence, $a:b=(m+\sqrt{m^2-n^2}):(m-\sqrt{m^2-n^2})$.

EXAMPLE 12 *Find two positive numbers whose difference is* 12 *and whose AM exceeds the GM by* 2.

SOLUTION Let the required positive numbers be a and b, where $a > b$.

Then, $a - b = 12$. … (i)

Also, $\text{AM} - \text{GM} = 2$

$$\Rightarrow \quad \frac{a+b}{2} - \sqrt{ab} = 2$$

$$\Rightarrow \quad a + b - 2\sqrt{ab} = 4$$

$$\Rightarrow \quad (\sqrt{a} - \sqrt{b})^2 = 4$$

$$\Rightarrow \quad \sqrt{a} - \sqrt{b} = 2. \qquad \ldots \text{(ii)}$$

Now, $a - b = 12$ [from (i)]

$$\Rightarrow \quad (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = 12$$

$$\Rightarrow \quad (\sqrt{a} + \sqrt{b}) \times 2 = 12 \Rightarrow \sqrt{a} + \sqrt{b} = 6. \qquad \ldots \text{(iii)} \quad [\text{using (ii)}]$$

On solving (ii) and (iii), we get $\sqrt{a} = 4$ and $\sqrt{b} = 2$.

$$\therefore \quad a = 16 \text{ and } b = 4.$$

Hence, the required numbers are 16 and 4.

EXAMPLE 13 *If* $x \in R$, *find the minimum value of* $3^x + 3^{(1-x)}$.

SOLUTION We know that $\text{AM} > \text{GM}$.

$\therefore$ for all $x \in R$, we have

$$\frac{3^x + 3^{(1-x)}}{2} > \sqrt{3^x \times 3^{(1-x)}}$$

$$\Rightarrow \quad \frac{3^x + 3^{(1-x)}}{2} > \sqrt{3}$$

$$\Rightarrow \quad \{3^x + 3^{(1-x)}\} > 2\sqrt{3}.$$

Hence, the minimum value of $3^x + 3^{(1-x)}$ is $2\sqrt{3}$ for any $x \in R$.

n GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS

Let a and b be two given numbers. We say that $G_1, G_2, G_3, \ldots, G_n$ *are n geometric means between a and b, if* $a, G_1, G_2, G_3, \ldots, G_n, b$ *are in GP.*

EXAMPLE 14 *Insert n geometric means between a and b.*

SOLUTION Let $G_1, G_2, G_3, \ldots, G_n$ be the n geometric means between a and b.

Then $a, G_1, G_2, G_3, \ldots, G_n, b$ are in GP.

This GP contains $(n+2)$ terms.

Let the common ratio of this GP be r. Then,

$$T_{n+2} = b \Rightarrow ar^{(n+1)} = b \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}.$$

$$\therefore \quad G_1 = ar = a \cdot \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}};$$

$$G_2 = ar^2 = a \cdot \left(\frac{b}{a}\right)^{\frac{2}{(n+1)}};$$

$$G_3 = ar^3 = a \cdot \left(\frac{b}{a}\right)^{\frac{3}{(n+1)}};$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$G_n = ar^n = a \cdot \left(\frac{b}{a}\right)^{\frac{n}{(n+1)}}.$$

EXAMPLE 15 *Insert two numbers between* 3 *and* 81 *so that the resulting sequence is a GP.*

SOLUTION Let the required numbers be G_1 and G_2. Then,

$3, G_1, G_2, 81$ are in GP.

Let the common ratio of this GP be r. Then,

$$T_4 = 81 \Rightarrow ar^{(4-1)} = 81$$
$$\Rightarrow 3 \times r^3 = 81 \Rightarrow r^3 = 27 = (3)^3 \Rightarrow r = 3.$$
$$\therefore \quad G_1 = (3 \times r) = (3 \times 3) = 9 \text{ and } G_2 = (3 \times r^2) = (3 \times 9) = 27.$$

Hence, the required numbers are 3 and 27.

EXAMPLE 16 *Insert three numbers between* 1 *and* 256 *so that the resulting sequence is a GP.*

SOLUTION Let the required numbers be G_1, G_2, G_3. Then,

$1, G_1, G_2, G_3, 256$ are in GP.

Let r be the common ratio of this GP. Then,

$$T_5 = 256 \Rightarrow ar^{(5-1)} = 256$$
$$\Rightarrow 1 \times r^4 = 256 \Rightarrow r^4 = 256 = (4)^4 \Rightarrow r = 4.$$
$$\therefore \quad G_1 = (1 \times 4) = 4, G_2 = (1 \times 4^2) = 16 \text{ and } G_3 = (1 \times 4^3) = 64.$$

Hence, the required numbers are 4, 16, 64.

EXAMPLE 17 *If G be the GM between two given numbers and A_1, A_2 be the two AMs between them, prove that*

$$G^2 = (2A_1 - A_2)(2A_2 - A_1).$$

SOLUTION Let a, b be the two given numbers. Then,

$$G^2 = ab. \qquad \ldots \text{(i)}$$

Also, a, A_1, A_2, b are in AP

$$\Rightarrow \quad 2A_1 = a + A_2 \text{ and } 2A_2 = A_1 + b$$
$$\Rightarrow \quad (2A_1 - A_2) = a \text{ and } (2A_2 - A_1) = b$$
$$\Rightarrow \quad (2A_1 - A_2)(2A_2 - A_1) = ab$$
$$\Rightarrow \quad (2A_1 - A_2)(2A_2 - A_1) = G^2 \quad \text{[using (i)]}.$$

Hence, $G^2 = (2A_1 - A_2)(2A_2 - A_1)$.

EXAMPLE 18 *If A is the arithmetic mean and G_1, G_2 be the two geometric means between any two numbers then prove that*

$$2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}.$$

SOLUTION Let the given numbers be a and b. Then,

$$A = \frac{a+b}{2} \Rightarrow a+b = 2A. \qquad \ldots (i)$$

And, a, G_1, G_2, b are in GP.

$$\therefore \quad \frac{G_1}{a} = \frac{G_2}{G_1} = \frac{b}{G_2}$$

$$\Rightarrow \quad a = \frac{G_1^2}{G_2} \text{ and } b = \frac{G_2^2}{G_1}$$

$$\Rightarrow \quad a+b = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$$

$$\Rightarrow \quad 2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} \qquad \text{[using (i)].}$$

Hence, $2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$.

EXERCISE 12F

1. Find two positive numbers a and b, whose
   (i) AM = 25 and GM = 20 (ii) AM = 10 and GM = 8.
2. Find the GM between the numbers
   (i) 5 and 125 (ii) 1 and $\frac{9}{16}$ (iii) 0.15 and 0.0015
   (iv) –8 and –2 (v) –6.3 and –2.8 (vi) a^3b and ab^3
3. Insert two geometric means between 9 and 243.
4. Insert three geometric means between $\frac{1}{3}$ and 432.
5. Insert four geometric means between 6 and 192.
6. The AM between two positive numbers a and $b(a > b)$ is twice their GM. Prove that $a : b = (2+\sqrt{3}) : (2-\sqrt{3})$.
7. If a, b, c are in AP; x is the GM between a and b; y is the GM between b and c; then show that b^2 is the AM between x^2 and y^2.
8. Show that the product of n geometric means between a and b is equal to the nth power of the single GM between a and b.
9. If AM and GM of the roots of a quadratic equation are 10 and 8 respectively then obtain the quadratic equation.

ANSWERS (EXERCISE 12F)

1. (i) (a = 40 and b = 10) or (a = 10 and b = 40)
   (ii) (a = 16 and b = 4) or (a = 4 and b = 16)
2. (i) 25 (ii) $\frac{3}{4}$ (iii) 0.015 (iv) –4 (v) –0.42 (vi) a^2b^2
3. 27, 81 **4.** 2, 12, 72 **5.** 12, 24, 48, 96 **9.** $x^2 - 20x + 64 = 0$

6. $\text{AM} = 2(\text{GM}) \Leftrightarrow \frac{1}{2}(a+b) = 2\sqrt{ab} \Leftrightarrow \frac{a+b}{2\sqrt{ab}} = \frac{2}{1}$

$$\Leftrightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{2+1}{2-1} \Leftrightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{(\sqrt{3})^2}{(1)^2}$$

$$\Leftrightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1} \Leftrightarrow \frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\Leftrightarrow \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \Leftrightarrow \frac{a}{b} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)^2} = \frac{2+\sqrt{3}}{2-\sqrt{3}}.$$

7. $(a+c) = 2b,\ x = \sqrt{ab}$ and $y = \sqrt{bc}$

$\Rightarrow \quad x^2 + y^2 = (ab+bc) = b(a+c) = 2b^2.$

8. Let $G_1, G_2, \ldots, G_n$ be n GMs between a and b, and let r be its common ratio. Then, $a, G_1, G_2, \ldots, G_n, b$ are in GP.

$$\therefore \quad T_{n+2} = b \Rightarrow ar^{(n+2-1)} = b \Rightarrow r^{(n+1)} = \left(\frac{b}{a}\right) \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}. \qquad \ldots \text{(i)}$$

$$\therefore \quad G_1 \times G_2 \times G_3 \times \ldots \times G_n = (ar) \times (ar^2) \times \ldots \times (ar^n)$$

$$= a^n \times r^{(1+2+\ldots+n)} = a^n \times r^{\frac{1}{2}n(n+1)}$$

$$= a^n \times \left(\frac{b}{a}\right)^{n/2} = a^{n/2} \times b^{n/2} \qquad \text{[using (i)]}$$

$$= (\sqrt{ab})^n = G^n, \text{ where } G = \sqrt{ab}.$$

9. Let a and b be the roots of the quadratic equation.

Then, $x^2 - (a+b)x + ab = 0.$

Now, $\frac{a+b}{2} = 10$ and $\sqrt{ab} = 8 \Rightarrow a+b = 20$ and $ab = 64.$

Hence, the required equation is $x^2 - 20x + 64 = 0.$

INFINITE GEOMETRIC SERIES

THEOREM *Prove that the sum of an infinite GP with first term a and common ratio r, where $|r| < 1$, is given by*

$$S = \frac{a}{(1-r)}.$$

PROOF Let us consider an infinite GP with first term a and common ratio r, where $-1 < r < 1$, i.e., $|r| < 1$.

The sum of n terms of a GP is given by

$$S_n = \frac{a(1-r^n)}{(1-r)} \Rightarrow S_n = \left\{\frac{a}{(1-r)} - \frac{ar^n}{(1-r)}\right\}. \qquad \ldots \text{(i)}$$

Since $|r| < 1$, so when n increases then r^n decreases.

Thus, when $n \to \infty$ then $r^n \to 0$.

$\therefore \quad \lim_{n\to\infty} \frac{r^n}{(1-r)} = 0.$

Hence, the sum of an infinite GP is given by

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left\{\frac{a}{(1-r)} - \frac{ar^n}{(1-r)}\right\}$$

$$= \frac{a}{(1-r)} - \lim_{n\to\infty} \frac{ar^n}{(1-r)} = \left\{\frac{a}{(1-r)} - 0\right\} = \frac{a}{(1-r)}$$

[using (i)].

This gives, $S = \frac{a}{(1-r)}$, when $|r| < 1$.

REMARKS 1. We shall denote the sum of an infinite GP by S.

2. If $r \geq 1$ then sum of an infinite GP is $S = \infty$.

SUMMARY

Sum of an infinite GP with the first term a and the common ratio r, where $|r| < 1$, is given by $S = \frac{a}{(1-r)}$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the sum of the infinite geometric series* $\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \infty\right)$.

SOLUTION In the given infinite geometric series, we have

$$a = 1 \text{ and } r = \frac{1}{3} \text{ such that } |r| = \frac{1}{3} < 1.$$

Hence, the sum of the given infinite series is

$$S = \frac{a}{(1-r)} = \frac{1}{\left(1 - \frac{1}{3}\right)} = \frac{1}{\left(\frac{2}{3}\right)} = \frac{3}{2}.$$

Hence, the required sum is $\frac{3}{2}$.

EXAMPLE 2 *Find the sum of the infinite geometric series* $\left(1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \ldots \infty\right)$.

SOLUTION The given series is an infinite geometric series in which

$$a = 1, r = -\frac{1}{3} \text{ and } |r| = \frac{1}{3} < 1.$$

Hence, the sum of the given infinite geometric series is

$$S = \frac{a}{(1-r)} = \frac{1}{\left(1 + \frac{1}{3}\right)} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}.$$

EXAMPLE 3 *Find the sum of the infinite geometric series* $\left(\frac{-5}{4} + \frac{5}{16} - \frac{5}{64} + \ldots \infty\right)$.

SOLUTION In the given infinite geometric series, we have

$$a = \frac{-5}{4}, r = \left(\frac{5}{16} \times \frac{4}{-5}\right) = -\frac{1}{4} \text{ and } |r| = \frac{1}{4} < 1.$$

Hence, the sum of the given infinite geometric series is

$$S = \frac{a}{(1-r)} = \frac{(-5/4)}{\left(1+\frac{1}{4}\right)} = \left(\frac{-5}{4} \times \frac{4}{5}\right) = -1.$$

EXAMPLE 4 *Find the sum of the infinite geometric series*

$$(\sqrt{2}+1)+1+(\sqrt{2}-1)+\ldots\infty.$$

SOLUTION We have

$$\frac{1}{(\sqrt{2}+1)} = \frac{1}{(\sqrt{2}+1)} \times \frac{(\sqrt{2}-1)}{(\sqrt{2}-1)} = \frac{(\sqrt{2}-1)}{1}.$$

So, the given series is an infinite geometric series in which $a = (\sqrt{2}+1)$ and $r = (\sqrt{2}-1) < 1$.

Hence, the sum of the given infinite geometric series is

$$S = \frac{a}{(1-r)} = \frac{(\sqrt{2}+1)}{\{1-(\sqrt{2}-1)\}} = \frac{(\sqrt{2}+1)}{(2-\sqrt{2})} \times \frac{(2+\sqrt{2})}{(2+\sqrt{2})}$$

$$= \frac{4+3\sqrt{2}}{(4-2)} = \frac{(4+3\sqrt{2})}{2}.$$

EXAMPLE 5 *Find the sum of the infinite series:*

$$\left\{\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\frac{1}{3^6}+\ldots\infty\right\}.$$

SOLUTION The given series can be expressed as the sum of two infinite geometric series, shown below.

$$\left\{\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\frac{1}{3^6}+\ldots\infty\right\}$$

$$= \left\{\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\ldots\infty\right\} + \left\{\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\ldots\infty\right\}$$

$$\left\{\text{an infinite GP with } a = \frac{1}{2} \text{ and } r = \frac{1}{4}\right\}$$

$$\left\{\text{an infinite GP with } a = \frac{1}{9} \text{ and } r = \frac{1}{9}\right\}$$

$$= \frac{(1/2)}{\left(1-\frac{1}{4}\right)} + \frac{(1/9)}{\left(1-\frac{1}{9}\right)} = \frac{(1/2)}{(3/4)} + \frac{(1/9)}{(8/9)}$$

$$= \left(\frac{1}{2} \times \frac{4}{3}\right) + \left(\frac{1}{9} \times \frac{9}{8}\right) = \left(\frac{2}{3} + \frac{1}{8}\right)$$

$$= \frac{16+3}{24} = \frac{19}{24}.$$

EXAMPLE 6 *Prove that* $6^{1/2} \cdot 6^{1/4} \cdot 6^{1/8} \cdot \ldots \infty = 6.$

SOLUTION We observe here that $\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty\right)$ is an infinite geometric series in which $a=\frac{1}{2}$ and $r=\left(\frac{1}{4}\times\frac{2}{1}\right)=\frac{1}{2}$ such that $|r|<1.$

So, this sum is given by

$$S=\frac{\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)}=\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}=1. \qquad \ldots \text{(i)}$$

$\therefore \quad 6^{\frac{1}{2}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{8}} \cdot \ldots \infty = 6^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty\right)} = 6^1 = 6$ [using (i)].

Hence, $6^{\frac{1}{2}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{8}} \cdot \ldots \infty = 6.$

EXAMPLE 7 *If* $|r|<1, x=\left(a+\frac{a}{r}+\frac{a}{r^2}+\ldots\infty\right), y=\left(b-\frac{b}{r}+\frac{b}{r^2}-\ldots\infty\right)$ *and* $z=\left(c+\frac{c}{r^2}+\frac{c}{r^4}+\ldots\infty\right)$, *prove that* $\frac{xy}{z}=\frac{ab}{c}.$

SOLUTION Clearly, each one of the given series is an infinite geometric series. Taking the sum of each series, we get

$$x=\frac{a}{\left(1-\frac{1}{r}\right)}=\frac{ar}{(r-1)} \quad \left[\because \text{1st term} = a \text{ and common ratio} = \frac{1}{r}\right],$$

$$y=\frac{b}{\left(1+\frac{1}{r}\right)}=\frac{br}{(r+1)} \quad \left[\because \text{1st term} = b \text{ and common ratio} = \frac{-1}{r}\right]$$

and $z=\frac{c}{\left(1-\frac{1}{r^2}\right)}=\frac{cr^2}{(r^2-1)} \quad \left[\because \text{1st term} = c \text{ and common ratio} = \frac{1}{r^2}\right]$

$$\therefore \quad \frac{xy}{z}=\frac{ar}{(r-1)}\times\frac{br}{(r+1)}\times\frac{(r^2-1)}{cr^2}=\frac{ab}{c}.$$

Hence, $\frac{xy}{z}=\frac{ab}{c}.$

EXAMPLE 8 *If* $y=x+x^2+x^3+\ldots\infty$, *where* $|x|<1$, *prove that* $x=\frac{y}{(1+y)}.$

SOLUTION By summing the given infinite GS, we get

$$y=\frac{x}{(1-x)} \Rightarrow y(1-x)=x$$

$$\Rightarrow y-xy=x \Rightarrow x+xy=y$$

$$x(1+y)=y \Rightarrow x=\frac{y}{(1+y)}.$$

Hence, $x=\frac{y}{(1+y)}.$

EXAMPLE 9 *If* $x = 1 + a + a^2 + \ldots \infty$, *where* $|a| < 1$
and $y = 1 + b + b^2 + \ldots \infty$, *where* $|b| < 1$,
prove that $(1 + ab + a^2b^2 + \ldots \infty) = \dfrac{xy}{(x + y - 1)}$.

SOLUTION By summing the given infinite GS, we get

$$x = \frac{1}{(1-a)} \text{ and } y = \frac{1}{(1-b)}.$$

$$\therefore \quad \frac{xy}{(x+y-1)} = \frac{\left\{\frac{1}{(1-a)} \cdot \frac{1}{(1-b)}\right\}}{\left\{\frac{1}{(1-a)} + \frac{1}{(1-b)} - 1\right\}}$$

$$= \left\{\frac{1}{(1-a)(1-b)} \times \frac{(1-a)(1-b)}{(1-b)+(1-a)-(1-a)(1-b)}\right\}$$

$$= \frac{1}{(1-ab)} = (1-ab)^{-1}$$

$$= (1 + ab + a^2b^2 + \ldots \infty).$$

Hence, $(1 + ab + a^2b^2 + \ldots \infty) = \dfrac{xy}{(x + y - 1)}$.

EXAMPLE 10 *If* $x = (1 + r^a + r^{2a} + \ldots \infty)$ *and* $y = (1 + r^b + r^{2b} + \ldots \infty)$, *prove that* $r = \left(\dfrac{x-1}{x}\right)^{\frac{1}{a}} = \left(\dfrac{y-1}{y}\right)^{\frac{1}{b}}$.

SOLUTION By summing the given infinite geometric series, we get

$$x = \frac{1}{(1-r^a)} \text{ and } y = \frac{1}{(1-r^b)}$$

$$\Rightarrow \quad (1-r^a) = \frac{1}{x} \text{ and } (1-r^b) = \frac{1}{y}$$

$$\Rightarrow \quad r^a = \left(1 - \frac{1}{x}\right) \text{ and } r^b = \left(1 - \frac{1}{y}\right)$$

$$\Rightarrow \quad r = \left(\frac{x-1}{x}\right)^{\frac{1}{a}} \text{ and } r = \left(\frac{y-1}{y}\right)^{\frac{1}{b}}.$$

Hence, $r = \left(\dfrac{x-1}{x}\right)^{\frac{1}{a}} = \left(\dfrac{y-1}{y}\right)^{\frac{1}{b}}$.

EXAMPLE 11 *Use geometric series to express* $0.555\ldots = 0.\overline{5}$ *as a rational number.*

SOLUTION We have

$$0.\overline{5} = 0.5555\ldots$$

$$= 0.5 + 0.05 + 0.005 + 0.0005 + \ldots \infty$$

$$= \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \frac{5}{10^4} + \ldots \infty$$

$$= \frac{\left(\frac{5}{10}\right)}{\left(1-\frac{1}{10}\right)} = \frac{5}{9} \qquad \left[\because\ S = \frac{a}{(1-r)} \text{ in an infinite GS}\right].$$

Hence, $0.\overline{5} = \frac{5}{9}$.

EXAMPLE 12 *Find the rational number whose decimal form is* $0.1\overline{42}$.

SOLUTION We have

$$0.1\overline{42} = 0.142424242\ldots.$$
$$= 0.1 + 0.042 + 0.00042 + 0.0000042 + \ldots\infty$$
$$= \frac{1}{10} + \left\{\frac{42}{10^3} + \frac{42}{10^5} + \frac{42}{10^7} + \ldots\infty\right\}$$
$$= \frac{1}{10} + \frac{\left(\frac{42}{10^3}\right)}{\left(1-\frac{1}{10^2}\right)} \quad \left[\text{this being a GS in which } a = \frac{42}{10^3} \text{ and } r = \left(\frac{42}{10^5}\times\frac{10^3}{42}\right) = \frac{1}{10^2} < 1\right]$$
$$= \frac{1}{10} + \left(\frac{42}{1000}\times\frac{100}{99}\right) = \left(\frac{1}{10} + \frac{42}{990}\right) = \frac{141}{990}.$$

Hence, $0.1\overline{42} = \frac{141}{990}$.

EXAMPLE 13 *Find the rational number whose decimal form is* $1.3\overline{45}$.

SOLUTION We have

$$1.3\overline{45} = 1.3454545\ldots\infty$$
$$= 1.3 + 0.045 + 0.00045 + \ldots\infty$$
$$= 1.3 + \left\{\frac{45}{10^3} + \frac{45}{10^5} + \ldots\infty\right\}$$
$$= 1.3 + \frac{\left(\frac{45}{10^3}\right)}{\left(1-\frac{1}{10^2}\right)} \quad \left[\text{this being a GS in which } a = \frac{45}{10^3} \text{ and } r = \frac{1}{10^2} < 1\right]$$
$$= 1.3 + \frac{45}{1000}\times\frac{100}{99} = \frac{13}{10} + \frac{45}{990} = \left(\frac{13}{10} + \frac{1}{22}\right)$$
$$= \frac{143+5}{110} = \frac{148}{110} = \frac{74}{55}.$$

Hence, $1.3\overline{45} = \frac{74}{55}$.

EXAMPLE 14 *The sum of an infinite GP is* $\frac{80}{9}$ *and its common ratio is* $\frac{-4}{5}$. *Find its first term.*

SOLUTION Let a be the first term of the given infinite GP.

Here, $r = \frac{-4}{5}$ and $|r| = \left|\frac{-4}{5}\right| = \frac{4}{5} < 1.$

The sum of this infinite GP is $S = \frac{80}{9}$.

$$\therefore \quad S = \frac{a}{(1-r)} \Rightarrow \frac{a}{\left(1+\frac{4}{5}\right)} = \frac{80}{9}$$

$$\Rightarrow \frac{5a}{9} = \frac{80}{9} \Rightarrow 5a = 80 \Rightarrow a = 16.$$

Hence, the first term is 16.

EXAMPLE 15 *The sum of first two terms of an infinite geometric series is* 15 *and each term of the series is equal to the sum of all the terms following it. Find the series.*

SOLUTION Let the series be $a + ar + ar^2 + \ldots + ar^n + ar^{(n+1)} + \ldots \infty$.

Then, first term = a and common ratio = r.

Clearly, $a + ar = 15 \Rightarrow a(1+r) = 15.$ … (i)

Also, $ar^{(n-1)} = ar^n + ar^{(n+1)} + \ldots \infty$ $[\because a_n = a_{n+1} + a_{n+2} + \ldots \infty]$

$$\Rightarrow \quad ar^{(n-1)} = \frac{ar^n}{(1-r)} \qquad \left[\because \begin{array}{l}\text{RHS is a GS in which first term} = ar^n \\ \text{and common ratio} = r\end{array}\right]$$

$$\Rightarrow \quad 1 - r = r \Rightarrow 2r = 1 \Rightarrow r = \frac{1}{2}.$$

Putting $r = \frac{1}{2}$ in (i), we get $\frac{3}{2}a = 15 \Rightarrow a = \left(15 \times \frac{2}{3}\right) = 10.$

Thus, $a = 10$ and $r = \frac{1}{2}$.

Hence, the required series is $\left(10 + 5 + \frac{5}{2} + \frac{5}{4} + \ldots \infty\right)$.

EXAMPLE 16 *The sum of an infinite geometric series is* 8. *If its second term is* 2, *find its common ratio.*

SOLUTION Let the infinite geometric series be $a, ar, ar^2, \ldots, \infty$. Then,

$$ar = 2 \qquad \ldots \text{(i)}$$

and $$\frac{a}{(1-r)} = 8. \qquad \ldots \text{(ii)}$$

Putting $r = \frac{2}{a}$ from (i) in (ii), we get

$$\frac{a}{\left(1-\frac{2}{a}\right)} = 8 \Rightarrow \frac{a^2}{(a-2)} = 8 \Rightarrow a^2 - 8a + 16 = 0$$

$$\Rightarrow (a-4)^2 = 0 \Rightarrow a - 4 = 0 \Rightarrow a = 4.$$

Putting $a = 4$ in (i), we get $r = \frac{2}{4} \Rightarrow r = \frac{1}{2}$.

Hence, the common ratio of the given geometric series is $\frac{1}{2}$.

EXAMPLE 17 *The sum of an infinite geometric series is* 15 *and the sum of the squares of these terms is* 45. *Find the series.*

SOLUTION Let a be the first term and r be the common ratio.

Then, the series is $a + ar + ar^2 + \ldots \infty$.

Now, $(a + ar + ar^2 + \ldots \infty) = 15 \Rightarrow \dfrac{a}{(1-r)} = 15$ … (i)

and $(a^2 + a^2r^2 + a^2r^4 + \ldots \infty) \Rightarrow \dfrac{a^2}{(1-r^2)} = 45.$ … (ii)

On squaring both sides of (i), we get $\dfrac{a^2}{(1-r)^2} = 225.$ … (iii)

On dividing (iii) by (ii), we get

$$\frac{a^2}{(1-r)^2} \times \frac{(1-r^2)}{a^2} = \frac{225}{45} \Rightarrow \frac{1+r}{1-r} = 5$$

$$\Rightarrow 1 + r = 5 - 5r$$

$$\Rightarrow 6r = 4 \Rightarrow r = \frac{2}{3}.$$

Putting $r = \dfrac{2}{3}$ in (i), we get

$$a = 15 \times \left(1 - \frac{2}{3}\right) = \left(15 \times \frac{1}{3}\right) = 5.$$

Thus, $a = 5$ and $r = \dfrac{2}{3}$.

Hence, the required series is $\left(5 + \dfrac{10}{3} + \dfrac{20}{9} + \dfrac{40}{27} + \ldots \infty\right)$.

EXAMPLE 18 *If $S_1, S_2, S_3, \ldots, S_p$ denote the sums of infinite geometric series whose first terms are $1, 2, 3, \ldots, p$ respectively and whose common ratios are $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots, \dfrac{1}{(p+1)}$ respectively, then show that*

$$S_1 + S_2 + S_3 + \ldots + S_p = \frac{1}{2}p(p+3).$$

SOLUTION By the summation of an infinite geometric series, we get

$$S_1 = \frac{1}{\left(1 - \frac{1}{2}\right)} = 2; S_2 = \frac{2}{\left(1 - \frac{1}{3}\right)} = 3; S_3 = \frac{3}{\left(1 - \frac{1}{4}\right)} = 4, \ldots,$$

$$S_p = \frac{p}{\left(1 - \frac{1}{p+1}\right)} = (p+1).$$

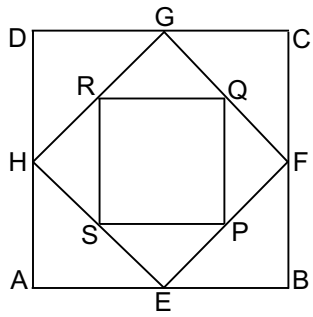
$\therefore$ $S_1 + S_2 + S_3 + \ldots + S_p = [2 + 3 + 4 + \ldots + (p+1)]$

$$= \frac{p}{2} \cdot \{2 + (p+1)\} = \frac{1}{2}p(p+3).$$

Hence, $S_1 + S_2 + S_3 + \ldots + S_p = \dfrac{1}{2}p(p+3)$.

EXAMPLE 19 *The side of a given square is 10 cm. The midpoints of its sides are joined to form a new square. Again, the midpoints of the sides of this new square are joined to form another square. This process is continued indefinitely. Find (i) the sum of the areas and (ii) the sum of the perimeters of the squares.*

SOLUTION Let *ABCD* be the given square with each side equal to 10 cm. Let *E, F, G, H* be the midpoints of the sides *AB, BC, CD* and *DA* respectively. Let *P, Q, R, S* be the midpoints of the sides *EF, FG, GH* and *HE* respectively.



$\therefore \quad BE = BF = 5 \text{ cm}$

$\Rightarrow \quad EF = \sqrt{BE^2 + BF^2} = \sqrt{25+25} \text{ cm}$

$= \sqrt{50} \text{ cm} = 5\sqrt{2} \text{ cm}.$

$\therefore \quad FQ = FP = \frac{1}{2}EF = \frac{5\sqrt{2}}{2} \text{ cm} = \frac{5}{\sqrt{2}} \text{ cm}$

$\Rightarrow \quad PQ = \sqrt{FP^2 + FQ^2} = \sqrt{\frac{25}{2} + \frac{25}{2}} \text{ cm} = \sqrt{25} \text{ cm} = 5 \text{ cm}.$

Thus, the sides of the squares are 10 cm, $5\sqrt{2}$ cm, 5 cm, … .

(i) Sum of the areas of the squares formed

$= \{(10)^2 + (5\sqrt{2})^2 + 5^2 + \ldots \infty\} \text{ cm}^2$

$= (100 + 50 + 25 + \ldots \infty) \text{ cm}^2$

$= \frac{100}{\left(1 - \frac{1}{2}\right)} \text{ cm}^2 = 200 \text{ cm}^2 \quad \left[\begin{array}{c}\text{taking the sum of infinite GP} \\ \text{with } a = 100 \text{ and } r = \frac{1}{2}\end{array}\right].$

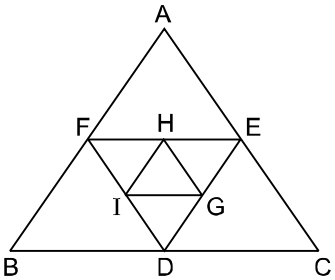
(ii) Sum of perimeters of the squares formed

$= (40 + 20\sqrt{2} + 20 + \ldots) \text{ cm}$

$= \frac{40}{\left(1 - \frac{1}{\sqrt{2}}\right)} \text{ cm} = \frac{40\sqrt{2}}{(\sqrt{2}-1)} \times \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)} \text{ cm} = (80 + 40\sqrt{2}) \text{ cm}.$

EXAMPLE 20 *Each side of an equilateral triangle is* 18 cm. *The midpoints of its sides are joined to form another triangle whose midpoints, in turn, are joined to form still another triangle. The process is continued indefinitely. Find the sum of the (i) perimeters of all the triangles and (ii) areas of all the triangles.*

SOLUTION Let $\triangle ABC$ be the given triangle having each side 18 cm. Let *D, E, F* be the midpoints of *BC, CA, AB* respectively to form $\triangle DEF$. Let *G, H, I* be the midpoints of *DE, EF* and *FD* respectively to form $\triangle GHI$.



We continue this process indefinitely.

The sides of these triangle are 18 cm, 9 cm, $\frac{9}{2}$ cm, …, and so on.

(i) Sum of the perimeters of all triangles so formed

$= 3\left\{18 + 9 + \frac{9}{2} + \frac{9}{4} + \ldots \infty\right\} \text{ cm}$

$= \left\{3 \times \frac{a}{(1-r)}\right\} \text{ cm, where } a = 18 \text{ and } r = \frac{9}{18} = \frac{1}{2}$

$$= \left\{3 \times \frac{18}{\left(1-\frac{1}{2}\right)}\right\} \text{cm} = (3 \times 36)\ \text{cm} = 108\ \text{cm}.$$

(ii) Sum of the areas of all the triangles so formed

$$= \frac{\sqrt{3}}{4} \cdot \left\{(18)^2 + (9)^2 + \left(\frac{9}{2}\right)^2 + \left(\frac{9}{4}\right)^2 + \ldots \infty\right\} \text{cm}^2$$

$$= \frac{\sqrt{3}}{4} \cdot \left\{324 + 81 + \frac{81}{4} + \frac{81}{16} + \ldots \infty\right\} \text{cm}^2$$

$$= \frac{\sqrt{3}}{4} \cdot \frac{a}{(1-r)}\ \text{cm}^2, \text{ where } a = 324 \text{ and } r = \frac{81}{324} = \frac{1}{4}$$

$$= \left\{\frac{\sqrt{3}}{4} \times \frac{324}{\left(1-\frac{1}{4}\right)}\right\} \text{cm}^2 = 108\sqrt{3}\ \text{cm}^2.$$

EXAMPLE 21 *After striking a floor a certain ball rebounds $\frac{4}{5}$ of the height from which it has fallen. Find the total distance that it travels before coming to rest, if it is gently dropped from a height of* 120 *metres.*

SOLUTION Initially, the ball falls from a height of 120 m. After striking the floor, it rebounds and goes to a height of $\left(\frac{4}{5} \times 120\right)$ m.

Now, it falls from this height and after striking the floor, it rebounds and goes to a height of $\frac{4}{5}$ of $\left(\frac{4}{5} \times 120\right)$ m $= \left\{\left(\frac{4}{5}\right)^2 \times 120\right\}$ m.

This process continues indefinitely till the ball comes to rest.

Total distance covered by the ball in metres

$$= 120 + 2 \times \left[\left\{\frac{4}{5} \times 120\right\} + \left\{\left(\frac{4}{5}\right)^2 \times 120\right\} + \left\{\left(\frac{4}{5}\right)^3 \times 120\right\} + \ldots \infty\right]$$

$$= 120 + \left\{2 \times \frac{a}{(1-r)}\right\}, \text{ where } a = \left(\frac{4}{5} \times 120\right) = 96 \text{ and } r = \frac{4}{5}$$

$$= \left\{120 + \frac{2 \times 96}{\left(1-\frac{4}{5}\right)}\right\} = \{120 + (192 \times 5)\} = (120 + 960) = 1080.$$

Hence, the total distance covered by the ball is 1080 m.

EXAMPLE 22 *If* $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$ *and* $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi$, *where* $0 < \theta < \phi < \frac{\pi}{2}$ *then prove that* $xz + yz - z = xy$.

SOLUTION We have

$$x = \sum_{n=0}^{\infty} \cos^{2n}\theta = 1 + \cos^2\theta + \cos^4\theta + \ldots \infty$$

$= \frac{a}{(1-r)}$, where $a = 1$ and $r = \cos^2\theta$ [sum of an infinite GP]

$= \frac{1}{(1-\cos^2\theta)} = \frac{1}{\sin^2\theta}$.

$\therefore \quad \sin^2\theta = \frac{1}{x}$. … (i)

$y = \sum_{n=0}^{\infty} \sin^{2n}\phi = 1 + \sin^2\phi + \sin^4\phi + \ldots \infty$

$= \frac{a}{(1-r)}$, where $a = 1$ and $r = \sin^2\phi$ [sum of an infinite GP]

$= \frac{1}{(1-\sin^2\phi)} = \frac{1}{\cos^2\phi}$.

$\therefore \quad \cos^2\phi = \frac{1}{y}$. … (ii)

$z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi = 1 + \cos^2\theta\sin^2\phi + \cos^4\theta\sin^4\phi + \ldots \infty$

$= \frac{a}{(1-r)}$, where $a = 1$ and $r = \cos^2\theta\sin^2\phi$

$= \frac{1}{(1-\cos^2\theta\sin^2\phi)} = \frac{1}{1-(1-\sin^2\theta)(1-\cos^2\phi)}$

$= \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} = \frac{1}{1-\left(1-\frac{1}{x}-\frac{1}{y}+\frac{1}{xy}\right)}$ [using (i) and (ii)]

$= \frac{1}{\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}\right)} = \frac{xy}{(x+y-1)}$.

$\therefore \quad z = \frac{xy}{x+y-1} \Rightarrow xz + yz - z = xy.$

Hence, $xz + yz - z = xy$.

EXERCISE 12G

Find the sum of each of the following infinite series:

1. $8 + 4\sqrt{2} + 4 + 2\sqrt{2} + \ldots \infty$

2. $6 + 1.2 + 0.24 + \ldots \infty$

3. $\sqrt{2} - \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}} + \ldots \infty$

4. $10 - 9 + 8.1 - \ldots \infty$

5. $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \ldots \infty$

6. Prove that $9^{1/3} \times 9^{1/9} \times 9^{1/27} \times \ldots \infty = 3$.

7. Find the rational number whose decimal expansion is given below:

(i) $0.\overline{3}$ (ii) $0.\overline{231}$ (iii) $3.5\overline{2}$

8. Express the recurring decimal $0.125125125\ldots = 0.\overline{125}$ as a rational number.

9. Write the value of $0.4\overline{23}$ in the form of a simple fraction.

10. Write the value of $2.1\overline{34}$ in the form of a simple fraction.

11. The sum of an infinite geometric series is 6. If its first term is 2, find its common ratio.

12. The sum of an infinite geometric series is 20 and the sum of the squares of these terms is 100. Find the series.

13. The sum of an infinite GP is 57 and the sum of their cubes is 9747. Find the GP.

ANSWERS (EXERCISE 12G)

1. $8(2+\sqrt{2})$ **2.** 7.5 **3.** $\frac{2\sqrt{2}}{3}$ **4.** $\frac{100}{19}$ **5.** $\frac{13}{24}$

7. (i) $\frac{1}{3}$ (ii) $\frac{231}{999}$ (iii) $\frac{317}{90}$ **8.** $\frac{125}{999}$ **9.** $\frac{419}{990}$ **10.** $\frac{2113}{990}$

11. $\frac{2}{3}$ **12.** $\left(8+\frac{24}{5}+\frac{72}{25}+\ldots\infty\right)$ **13.** $\left(19, \frac{38}{3}, \frac{76}{9}, \frac{152}{27}, \ldots\infty\right)$

HINTS TO SOME SELECTED QUESTIONS

1. $a = 8$ and $r = \frac{4\sqrt{2}}{8} = \frac{1}{\sqrt{2}}$.

$\therefore \quad S = \frac{a}{(1-r)} = \frac{8}{\left(1-\frac{1}{\sqrt{2}}\right)} = \frac{8\sqrt{2}}{(\sqrt{2}-1)} \times \frac{(\sqrt{2}+1)}{(\sqrt{2}+1} = 8(2+\sqrt{2})$.

2. $a = 6$ and $r = \frac{1.2}{6} = 0.2$.

$\therefore \quad S = \frac{a}{(1-r)} = \frac{6}{0.8} = \frac{60}{8} = \frac{15}{2} = 7.5$.

3. $a = \sqrt{2}$ and $r = \frac{-1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{-1}{2}$.

$\therefore \quad S = \frac{a}{(1-r)} = \frac{\sqrt{2}}{\left(1+\frac{1}{2}\right)} = \frac{2\sqrt{2}}{3}$.

4. $a = 10$ and $r = \frac{-9}{10} = -0.9$.

$\therefore \quad S = \frac{a}{(1-r)} = \frac{10}{(1+0.9)} = \frac{10}{1.9} = \frac{100}{19}$.

5. $S = \left(\frac{2}{5}+\frac{2}{5^3}+\frac{2}{5^5}+\ldots\infty\right)+\left(\frac{3}{5^2}+\frac{3}{5^4}+\frac{3}{5^6}+\ldots\infty\right)$

$= \frac{a_1}{(1-r_1)}+\frac{a_2}{(1-r_2)}$, where $\left(a_1 = \frac{2}{5}, r_1 = \frac{1}{25}\right)$ and $\left(a_2 = \frac{3}{25}, r_2 = \frac{1}{25}\right)$.

6. $9^{1/3} \times 9^{1/9} \times 9^{1/27} \times \ldots \infty$

$= 9^{\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\infty\right)} = 9^{\frac{a}{(1-r)}}$, where $a = \frac{1}{3}$ and $r = \frac{1}{3}$

$= 9^{\frac{(1/3)}{\left(1-\frac{1}{3}\right)}} = 9^{\frac{1}{2}} = \sqrt{9} = 3.$

7. (ii) $x = 0.231231231\ldots$ and $1000x = 231.231231231\ldots$.

On subraction, we get $999x = 231 \Rightarrow x = \frac{231}{999}$.

Hence, $0.\overline{231} = \frac{231}{999}$.

(iii) $x = 3.5222\ldots \Rightarrow 10x = 35.222\ldots$ and $100x = 352.222\ldots$.

$\therefore \quad 90x = 317 \Rightarrow x = \frac{317}{90}$.

8. $x = 0.125125\ldots \Rightarrow 1000x = 125.125125\ldots$.

$\therefore \quad 999x = 125 \Rightarrow x = \frac{125}{999}$.

Hence, $0.125125\ldots = \frac{125}{999}$.

9. $x = 0.42323\ldots \Rightarrow 10x = 4.232323\ldots$ and $1000x = 423.2323\ldots$.

$\therefore \quad 990x = (423 - 4) = 419 \Rightarrow x = \frac{419}{990}$.

Hence, $0.4\overline{23} = \frac{419}{990}$.

10. $x = 2.1343434\ldots \Rightarrow 10x = 21.343434\ldots$ and $1000x = 2134.3434\ldots$.

$\therefore \quad 990x = (2134 - 21) = 2113 \Rightarrow x = \frac{2113}{990}$.

Hence, $2.1\overline{34} = \frac{2113}{990}$.

11. $\frac{a}{1-r} = 6 \Rightarrow \frac{2}{1-r} = 6 \Rightarrow 1 - r = \frac{1}{3} \Rightarrow r = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$.

12. Let a be the first term and r be the common ratio of the given series.

Then, $\frac{a}{(1-r)} = 20 \Rightarrow \frac{a^2}{(1-r)^2} = 400.$ … (i)

And, $\frac{a^2}{(1-r^2)} = 100.$ … (ii)

$\therefore \quad \frac{a^2}{(1-r)^2} \times \frac{(1-r^2)}{a^2} = \frac{400}{100} \Rightarrow \frac{1+r}{1-r} = 4 \Rightarrow r = \frac{3}{5}$.

$\therefore \quad \frac{a}{\left(1-\frac{3}{5}\right)} = 20 \Rightarrow a = 8.$

Hence, the required series is $\left(8 + \frac{24}{5} + \frac{72}{25} + \ldots \infty\right)$.

13. Let a be the first term and r be the common ratio of the given GP.

Then, $\frac{a}{1-r} = 57 \Rightarrow \frac{a^3}{(1-r)^3} = (57)^3.$ … (i)

And, $a^3 + a^3r^3 + a^3r^6 + \ldots \infty = 9747$

$\Rightarrow \quad \dfrac{a^3}{(1-r^3)} = 9747.$ … (ii)

$\therefore \quad \dfrac{a^3}{(1-r)^3} \times \dfrac{(1-r^3)}{a^3} = \dfrac{(57)^3}{9747} \Rightarrow \dfrac{(1-r^3)}{(1-r)^3} = 19 \Rightarrow \dfrac{1+r+r^2}{(1-r)^2} = 19$

$\Rightarrow 1 + r + r^2 = 19r^2 - 38r + 19$

$\Rightarrow 18r^2 - 39r + 18 = 0$

$\Rightarrow 18r^2 - 27r - 12r + 18 = 0$

$\Rightarrow (3r-2)(6r-9) = 0 \Rightarrow r = \dfrac{2}{3}$

$\left[\because\ r \neq \dfrac{3}{2}, \text{ as } -1 < r < 1 \text{ for infinite GP}\right].$

$\therefore \quad \dfrac{a}{\left(1-\frac{2}{3}\right)} = 57 \Rightarrow a = \left(57 \times \dfrac{1}{3}\right) = 19.$

Hence, the required GP is $19, \dfrac{38}{3}, \dfrac{76}{9}, \dfrac{152}{27}, \ldots \infty.$

EXERCISE 12H

Very-Short-Answer Questions

1. If the 5th term of a GP is 2, find the product of its first nine terms.
2. If the $(p+q)$th and $(p-q)$th terms of a GP are m and n respectively, find its pth term.
3. If 2nd, 3rd and 6th terms of an AP are the three consecutive terms of a GP then find the common ratio of the GP.
4. Write the quadratic equation, the arithmetic and geometric means of whose roots are A and G respectively.
5. If a, b, c are in GP and $a^{1/x} = b^{1/y} = c^{1/z}$ then prove that x, y, z are in AP.
6. If a, b, c are in AP and x, y, z are in GP then prove that the value of $x^{b-c} \cdot y^{c-a} \cdot z^{a-b}$ is 1.
7. Prove that $\left(1 - \dfrac{1}{3} + \dfrac{1}{3^2} - \dfrac{1}{3^3} + \dfrac{1}{3^4} - \ldots \infty\right) = \dfrac{3}{4}.$
8. Express $0.\overline{123}$ as a rational number.
9. Express $0.\overline{6}$ as a rational number.
10. Express $0.6\overline{8}$ as a rational number.
11. The second term of a GP is 24 and its fifth term is 81. Find the sum of its first five terms.
12. The ratio of the sum of first three terms is to that of first six terms of a GP is 125 : 152. Find the common ratio.
13. The sum of first three terms of a GP is $\dfrac{39}{10}$ and their product is 1. Find the common ratio and these three terms.

ANSWERS (EXERCISE 12H)

1. 512 **2.** $\sqrt{mn}$ **3.** 3 **4.** $x^2 - 2Ax + G^2 = 0$

8. $\frac{123}{999}$ **9.** $\frac{2}{3}$ **10.** $\frac{31}{45}$ **11.** 211 **12.** $\frac{3}{5}$

13. $r = \frac{5}{2}$ or $\frac{2}{5}$; $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$

HINTS TO SOME SELECTED QUESTIONS

1. Let first term $= a$ and common ratio $= r$. Then,

$T_5 = 2 \Rightarrow ar^4 = 2.$

Required product

$= (a \times ar \times ar^2 \times \ldots \times ar^8)$

$= a^9 \times r^{(1+2+3+\ldots+8)} = a^9 r^{36} = (ar^4)^9 = 2^9 = 512.$

2. Let first term $= a$ and common ratio $= r$. Then,

$T_{p+q} = m \Rightarrow ar^{(p+q-1)} = m$

$T_{p-q} = n \Rightarrow ar^{(p-q-1)} = n$

$\therefore \quad mn = a^2 r^{2(p-1)} = \{ar^{(p-1)}\}^2 \Rightarrow ar^{(p-1)} = \sqrt{mn} \Rightarrow T_p = \sqrt{mn}.$

3. $(a+d), (a+2d), (a+5d)$ are in GP.

$\therefore \quad \frac{a+2d}{a+d} = \frac{a+5d}{a+2d} \Rightarrow (a+2d)^2 = (a+d)(a+5d)$

$\Rightarrow d^2 + 2ad = 0 \Rightarrow d(d+2a) = 0 \Rightarrow d = -2a \qquad [\because d \neq 0].$

So, these terms are $-a, -3a, -9a$. Its common ratio $= \frac{-3a}{-a} = 3.$

4. Let the roots of the quadratic equation be α and β.

Then, the equation is $x^2 - (\alpha + \beta)x + \alpha\beta = 0.$

$\therefore \quad A = \frac{\alpha + \beta}{2} \Rightarrow \alpha + \beta = 2A.$

$G = \sqrt{\alpha\beta} \Rightarrow \alpha\beta = G^2.$

Hence, the required equation is $x^2 - 2Ax + G^2 = 0.$

5. Let $a^{1/x} = b^{1/y} = c^{1/z} = m$. Then,

$a = m^x, b = m^y$ and $c = m^z.$

Since a, b, c are in GP, we have

$b^2 - ac \Rightarrow (m^y)^2 - (m^x \times m^z)$

$\Rightarrow m^{2y} = m^{x+z} \Rightarrow 2y = x + z.$

$\therefore \quad x, y, z$ are in AP.

6. a, b, c are in AP $\Rightarrow 2b = a + c.$

x, y, z are in GP $\Rightarrow y^2 = xz.$

$\therefore \quad x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)}$

$= x^{(b-c)} \cdot (\sqrt{xz})^{(c-a)} \cdot z^{(a-b)} \qquad [\because y = \sqrt{xz}]$

$$= x^{(b-c)} \cdot x^{\frac{1}{2}(c-a)} \cdot z^{\frac{1}{2}(c-a)} \cdot z^{(a-b)}$$

$$= x^{(b-c)+\frac{1}{2}(c-a)} \cdot z^{\frac{1}{2}(c-a)+(a-b)}$$

$$= x^{\frac{1}{2}\{2b-(a+c)\}} \cdot z^{\frac{1}{2}\{c+a-2b\}} = x^0 \times z^0 = 1.$$

7. This is an infinite GP in which $a = 1$ and $r = \frac{-1}{3}$.

$$S_\infty = \frac{a}{(1-r)} = \frac{1}{\left(1+\frac{1}{3}\right)} = \frac{3}{4}.$$

8. $x = 0.123123\ldots$ and $1000x = 123.123123\ldots$.

$$\therefore \quad 999x = 123 \Rightarrow x = \frac{123}{999}.$$

10. $x = 0.688\ldots \Rightarrow 10x = 6.888\ldots$ and $100x = 68.888\ldots$.

$$\therefore \quad 90x = 62 \Rightarrow x = \frac{62}{90} = \frac{31}{45}.$$

11. $ar = 24$ and $ar^4 = 81$.

$$\therefore \quad \frac{ar^4}{ar} = \frac{81}{24} = \frac{27}{8} \Rightarrow r^3 = \frac{27}{8} = \left(\frac{3}{2}\right)^3 \Rightarrow r = \frac{3}{2}.$$

$$\therefore \quad a \times \frac{3}{2} = 24 \Rightarrow a = \left(24 \times \frac{2}{3}\right) = 16.$$

$$S_5 = \frac{a(r^5-1)}{(r-1)} = \frac{16 \times \left\{\left(\frac{3}{2}\right)^5 - 1\right\}}{\left(\frac{3}{2}-1\right)} = \frac{32 \times (3^5 - 2^5)}{2^5} = (243 - 32) = 211.$$

12. $\frac{a(r^3-1)}{(r-1)} \times \frac{(r-1)}{a(r^6-1)} = \frac{125}{152} \Rightarrow \frac{1}{(r^3+1)} = \frac{125}{152} \Rightarrow (r^3+1) = \frac{152}{125}.$

$$\therefore \quad r^3 = \left(\frac{152}{125} - 1\right) = \frac{27}{125} = \left(\frac{3}{5}\right)^3 \Rightarrow r = \frac{3}{5}.$$

13. Let the three terms of the GP be $\frac{a}{r}$, a and ar. Then,

$$\frac{a}{r} \times a \times ar = 1 \Rightarrow a^3 = 1 \Rightarrow a = 1.$$

So, the given terms are $\frac{1}{r}$, 1 and r.

$$\therefore \quad \frac{1}{r} + 1 + r = \frac{39}{10} \Rightarrow r + \frac{1}{r} = \left(\frac{39}{10} - 1\right) = \frac{29}{10}$$

$$\Rightarrow 10r^2 + 10 = 29r$$

$$\Rightarrow 10r^2 - 29r + 10 = 0$$

$$\Rightarrow (2r-5)(5r-2) = 0$$

$$\Rightarrow r = \frac{5}{2} \text{ or } r = \frac{2}{5}.$$

Hence, these terms are $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$.

SUMMARY OF FACTS AND FORMULAE

1. (i) A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called a geometrical progression (abbreviated as GP), if each term is nonzero and $\frac{a_{k+1}}{a_k} = r$ (constant) for all $k \geq 1$.

The constant ratio $= r$ is called the common ratio.

(ii) In a GP, we denote the first term by a, common ratio by r and nth term by T_n.

(iii) We have, $T_n = ar^{n-1}$.

(iv) nth term from the end of a GP with first term a, common ratio r and the last term l is $\frac{l}{r^{(n-1)}}$.

2. For solving problems on GP, it is convenient to take

(i) 3 numbers in GP as $\frac{a}{r}, a, ar$.

(ii) 4 numbers in GP as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

(iii) 5 numbers in GP as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$.

(iv) $a, ar, ar^2, ar^3, \ldots$ when their product is not given.

3. Sum of n terms of a GP with first term $= a$ and common ratio $= r$ is given by

$$S_n = \begin{cases} na, \text{ when } r = 1 \\ \frac{a(1-r^n)}{(1-r)}, \text{ when } r < 1 \\ \frac{a(r^n-1)}{(r-1)}, \text{ when } r > 1. \end{cases}$$

4. The sum of an infinite GP with first term $= a$ and common ratio $= r$, where $|r| < 1$ is given by $S = \frac{a}{(1-r)}$.

5. Properties of GP

Let $a, ar, ar^2, ar^3, \ldots$ be a GP. Then,

(i) $\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \frac{1}{ar^3}, \ldots$ is a GP with first term $= \frac{1}{a}$ and common ratio $= \frac{1}{r}$.

(ii) $ka, k(ar), k(ar^2), k(ar^3), \ldots$ is a GP (where $k \neq 0$) with first term $= ka$ and common ratio $= r$.

(iii) $\frac{k}{a}, \frac{k}{ar}, \frac{k}{ar^2}, \frac{k}{ar^3}, \ldots$ is a GP (where $k \neq 0$) with first term $= \frac{k}{a}$ and common ratio $= \frac{1}{r}$.

(iv) $a^k, (ar)^k, (ar^2)^k, \ldots$ is a GP (where $k \neq 0$) with first term $= a^k$ and common ratio $= r^k$.